Limited enforcement, bubbles and trading in incomplete markets *

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Abstract

We show that a large class of rational bubbles relax consumers' debt limits. The collapse of a bubble amounts to a contraction of credit, and conversely, a bubble can arise to supplement the credit available in the economy. As a by-product, however, bubbles can cause large increases in trading volume, volatile asset prices and high and time-varying Sharpe ratios.

Keywords: rational bubbles, trading volume, credit crunch, endogenous debt limits, limited enforcement, excess volatility, conditional and unconditional equity premium puzzle

1 Introduction

Episodes of large stock market run-ups followed by abrupt crashes, without apparent similar movements in fundamentals, are referred to as bubbles. They are typically accompanied by large increases in trading volume (Cochrane 2002). Formally, a (rational) bubble is defined

*Some of the results in here are based on Chapter 3 in Bidian (2011).

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as the price of an asset in excess of its fundamental value, computed as the discounted (at market rates) present value of dividends.

We show that an intrinsic property of a large class of rational bubbles is their capacity to relax the debt limits of the agents. Any bubble that preserves the set of deflators (or the asset span) is effectively equivalent (from the point of view of allocations) with an appropriate relaxation of debt limits, proportional with the size of the bubble. Thus the collapse of a bubble amounts to a contraction of agents' debt limits, and conversely, a bubble can arise to supplement the credit available in the economy. Even when comparing a bubbly economy to the equivalent bubble-free economy with relaxed debt limits, bubbles distort trading volumes and asset returns.

We build on the insight of Kocherlakota (2008), who showed that arbitrary discounted (by the pricing kernel) positive martingales can be introduced into asset prices as bubbles, while leaving agents' consumption and the pricing kernel unchanged, as long as the debt constraints of the agents are allowed to be adjusted upwards by their initial endowment of the assets multiplied with the bubble term. The introduction of a bubble gives consumers a windfall proportional to their initial holding of the asset, which can be sterilized, leaving their budgets unaffected, by an appropriate tightening of the debt limits. The modified debt constraints bind in exactly the same dates and states. Kocherlakota (2008) refers to this result as "the bubble equivalence theorem", and to this technique of introducing bubbles as "bubble injections".

A major limitation of Kocherlakota's (2008) result is the assumption that agents can trade in a full set of state-contingent claims to consumption next period, in addition to the existing long-lived securities. Hence one might infer that bubble injections are associated to knife-edge situations, and they might not apply to incomplete markets environments or even to economies with dynamically complete markets (rather than Arrow-Debreu complete). Moreover, as explained below, in the presence of redundant assets, the effect of bubble injections on trading volumes is not a well-posed problem. Therefore his result cannot justify the increases in stock market trading volume associated with the presence of bubbles.

We prove that the bubble equivalence theorem holds even when markets are incomplete, or only dynamically complete. We characterize completely the set of processes that can be injected as bubbles in asset prices. We show that any process with a positive rate of growth equal to the gross return of some trading strategy can be introduced as a bubble in asset prices, if it does not decreases the asset span. This "drop in rank" of the asset span does not occur generically, therefore the set of potential bubbles is very large. The result works also in reverse, generically any economy having a bubble (in some asset) which has a rate of growth equal to the return of a trading strategy is equivalent with a bubble-free economy if debt limits are relaxed in proportion to the size of the bubble.

It follows that, for general environments, an unexpected bubble collapse would not affect agents' consumption if their debt limits are relaxed by an amount proportional to the size of the bubble. In the absence of such an increase in the availability of credit, a bubble collapse amounts to a credit crunch, and therefore can be contractionary (see, for example, Guerrieri and Lorenzoni 2011).

The bubble equivalence theorem has additional appeal in environments with endogenous debt limits, as in Alvarez and Jermann (2000). In these models, agents have the option to default on debt and receive a predetermined continuation utility, and the markets (competitive financial intermediaries) select the largest debt limits so that repayment is always individually rational given future bounds on debt. It turns out that the tighter debt limits required to sustain a bubble injection are again the endogenous bounds allowing for maximal credit expansion and preventing default. We allow for more general punishments after default than in Kocherlakota (2008). In particular, we cover the case where upon default the agents are forbidden to carry debt (Bulow and Rogoff 1989, Hellwig and Lorenzoni 2009).

We analyze next the trading volume effects of bubbles. In light of the bubble equivalence theorem, it is not surprising that a bubbly economy allows for more trade than an economy without bubbles and identical debt limits, as the bubble effectively expands the debt limits of the agents and increases the risk-sharing opportunities. However, we compare the trading volumes in an economy with bubbles and the equivalent (from the point of view of allocations) bubble-free economy with more relaxed debt limits. In this comparison, bubbles are kept "neutral" from the point of view of allocations, but nevertheless bubbly economies can display larger trading volumes, as agents adjust their portfolios in response to a bubble injection. Therefore even bubble injections in zero-supply assets, which create no wealth effects and do not affect agents' debt limits, can generate increases in trading volume. Such portfolio effects are impossible to quantify unambiguously in Kocherlakota (2008), as bubble-carrying stocks are redundant assets. Indeed, one can arrange the equilibrium portfolios such that there is no additional trade in the redundant stocks after a bubble injection and all portfolio adjustments are done via Arrow securities (this is what Kocherlakota (2008) does), but equivalently, one could have arbitrary amounts of trading in stocks by appropriately adjusting the holdings of Arrow securities.

In Section 2, we illustrate the expansionary effects of bubbles, as well as their effect on trading volume, in a Bewley style, deterministic economy, with elastic labor supply. The example is an extension of the one analyzed in Kocherlakota (2011), or alternatively, can be viewed as a tractable particular case of the model studied computationally in Guerrieri and Lorenzoni (2011). We show that a tightening of the debt limits results in a contraction of output, consumption, real interest rates, trade and welfare. When the agents have access to fiat money, in addition to Arrow securities, a bubble (valued money) of appropriate size matching the size of the contraction in debt limits effectively prevents the recessionary effects of the credit crunch. The bubble creates no additional trade in the Arrow securities and in money, compared to the equilibrium without the bubble and initial (pre-credit crunch) debt limits. By replacing the fiat money with a dividend paying asset (hence with non-zero intrinsic value), the Arrow securities are not needed anymore to enable trade in the absence of a bubble, as the asset dynamically completes the markets. Now the readjustment in agents' portfolios in response to the bubble injection leads to increases in the trading volume. When the dividends of the asset go to zero, thus approaching the fiat money case, the volume of trade doubles (in the limit).

A bubble injection unambiguously increases the trading volume in dynamically complete markets economies with Markov endowments and non-binding borrowing constraints. In these environments, Judd, Kubler, and Schmedders (2003) show that there is no trade in stocks (for a generic set of dividends), after an initial portfolio rebalancing. A bubble in one asset now distorts agents' portfolios and can create trading in all of the assets. Trade can be absent in a bubble-free equilibrium in environments more general than those covered by Judd, Kubler, and Schmedders (2003), as long as the equilibrium remains Pareto optimal (hence the debt limits do not bind). We illustrate this in the example in Section C, where we allow for incomplete markets and alternative debt limits, and show that a bubble creates persistent (bounded away from zero) trading volume.

For a quantitative exploration of the trading volume effects of bubble injections, and to show that volume effects are also present when debt constraints bind, we consider the limited enforcement economy of Alvarez and Jermann (2001). They make a compelling empirical case for the improved asset pricing performance of models with limited enforcement in a stochastic economy with two-agents, complete markets and an interdiction to trade as penalty for default. We substitute the Arrow securities with long-lived assets that dynamically complete the markets and show that bubbles (deterministic or stochastic) increase the trading volume. Small initial bubbles (0.1% of GDP) can generate large increases in trading volume (up to 8% of GDP) each period while they run. They also generate large drops in trade volume upon their collapse. When the bubble crashes after a long run, the volume of trade reduces roughly by half (drop of 16% of GDP). There are "contagion" effects, as a bubble in one asset boosts the trading volume also in the other asset.

A bubble injection in an asset also distorts the price and the return on the asset, which becomes a weighted average between the fundamental return of the asset and the bubble growth rate. The weight of the fundamental return in the bubbly return is just the ratio of fundamental value to the bubble inflated price. We show that bubbles can help explain the "excess volatility puzzle" - the large volatility of asset prices, with little movements in dividends or consumption. Moreover, a bubble in an asset increases the conditional expected risk premium, respectively conditional Sharpe ratio of the asset if the rate of growth of the bubble *covaries*, respectively is *correlated more negatively* to the stochastic discount factor than the fundamental asset return. In fact appropriately chosen bubbles can make the conditional Sharpe ratio (or equity premium, or kurtosis) of an asset return be arbitrarily close to the maximal one attainable by a portfolio with positive gross returns. Bubble injections can also generate time-varying (and countercyclical) conditional risk premia and Sharpe ratios, in line with the documented "conditional equity premium puzzle" (Cochrane 2000, Chapter 21).

Rational bubbles do not have to be nonstationary, as commonly believed, and therefore at odds with empirical observations. With low interest rates, bubbles can grow at the rate of aggregate endowment and be stationary. This can be seen in the example in Section 2, or in some of the examples in Hellwig and Lorenzoni (2009) and Bidian and Bejan (2012). Bidian (2011, Chapter 5) analyzes in detail the rather loose connection between the existence of bubbles and the stationarity properties of the dividend yield. It should be also emphasized that there is no contradiction between the possibility of bubble injections discussed here and the nonexistence of bubbles on positive supply assets in economies with "high interest rates", that is with finite present value of aggregate consumption (Santos and Woodford 1997, Huang and Werner 2000, Kocherlakota 1992).[1] The nonexistence of bubbles results rely on the hidden assumption that the debt limits faced by agents are nonpositive, while the adjusted debt bounds after a bubble injection must become positive

[1]Bubbles grow on average at the rate of interest rates. With high interest rates, the bubble must become very large relative to aggregate endowment, even if this happens with small probability. But this is incompatible with the presence of optimizing, forward looking agents, who do not allow their financial wealth to exceed the present value of their future consumption.

eventually, if the asset is in positive supply and the interest rates are *high*, even though this may happen with arbitrarily small probability. However, *low* interest rates are the natural result of the existence of enforcement limitations, since in equilibrium the interest rates adjust to a lower level to entice agents to repay their debt and prevent default. Bidian and Bejan (2012) show that bubble injections leading to nonpositive debt limits are possible for the most common types of penalties for default encountered in the literature: a permanent, respectively temporary interdiction to trade, or an interdiction to borrow.

To our knowledge, our paper is the first that shows that rational bubbles can generate increases in trading volume. There is a well developed literature on *speculative bubbles* in economies with short sale constraints and heterogeneous beliefs (Harrison and Kreps 1978, Morris 1996, Scheinkman and Xiong 2003), which arose in response to the difficulty of generating rational bubbles (the nonexistence results mentioned before), and because (Scheinkman and Xiong 2003)

> "rational bubble models are incapable of connecting bubbles with turnover."

The cited papers use partial equilibrium models, in which infinitely wealthy risk neutral agents pass the asset from one to another, the pessimists selling it to the optimists. Beliefs are constructed such that agents take turns in being the optimists, which results in frequent trading. In the speculative bubbles literature, the price of the asset is in fact equal with the present value of its dividends discounted at market rates, so the presence of overvaluation is debatable.[2] Additionally, if learning is allowed, agents' beliefs converge and the bubble disappears generically (Morris 1996, Slawski 2008). As explained in Scheinkman and Xiong (2003), it is also hard to generate realistic time series dynamics for speculative bubbles.[3] Furthermore, these very stylized models are unable to connect bubbles to macro aggregates such as consumption, output, interest rates.

[2]In defining a speculative bubble, the fundamental value of an asset is taken to be the maximum amount that an agent would be willing to pay when forced to maintain the holdings of the asset forever. Thus the dividends of the asset are discounted using the same agent's intertemporal marginal rates of substitution (IMRS), rather than using the marginal's agent (the buyer of the asset) IMRS. Thus a speculative bubble is actually the "convenience yield" accruing to an agent holding an asset subject to short sale constraints. Indeed, with shorting restrictions, an agent that keeps inventories of an asset has the option to sell it if its price is high and can better smooth demand shocks, and therefore enjoys a convenience yield (Cochrane 2002). Duffie, Garleanu, and Pedersen (2002) rationalize the convenience yield induced by short sale constraints as the value of lending fees arising from searching for security lenders and bargaining over the terms of lending.

[3]Run-ups in asset prices would require a continual increase in the dispersion of beliefs, and bubble collapses would require a sudden alignment of beliefs.

In our paper, bubbles enable agents to circumvent a reduction in the availability of credit, and to achieve identical allocations to those possible under more relaxed, but still self-enforcing debt limits. A host of recent papers point out, similarly, that bubbles can arise in the presence of financial frictions, and help relax the underlying borrowing constraints (Kocherlakota 2009, Martin and Ventura 2012, Giglio and Severo 2012, Farhi and Tirole 2012). Those bubbles facilitate the transfer of resources from unproductive entrepreneurs to the productive ones, by increasing the borrowing capacity of the latter. Miao and Wang (2011)[4] make a related point, but they emphasize the multiplicity of equilibria in economies with limited enforcement, studied also in Hellwig and Lorenzoni (2009) and Bidian and Bejan (2012). None of these papers can address the trading volume effects of bubbles.

Debt is fully collateralized (secured) by a (pleadgeable) fraction of the capital in Farhi and Tirole (2012), by the physical (but not the intangible) capital in Giglio and Severo (2012)), or by bubbles in Kocherlakota (2009) and Martin and Ventura (2012). These papers also focus on the production sector, shutting down (non-entrepreneurs) consumers from borrowing and lending.[5] By contrast, our model allows for unsecured debt, sustained solely by reputation (for example, credit card debt). We focus squarely on the consumer sector, allowing consumers to borrow and lend to each other, in a Bewley-Aiyagari environment with infinitely lived agents. As argued in Guerrieri and Lorenzoni (2011) and the references therein, the response of the consumer (household) sector to the credit tightening is crucial in explaining the recent U.S. recession. Our paper indicates that the collapse of the large housing bubble was tantamount to a (proportionally) large credit crunch, whose contractionary effects can lead to the experienced recession.

The paper is organized as follows. Section 2 contains an illustrative example. Section 3 presents the model and the bubble equivalence theorem. Section 4 analyzes the implications of bubble injections on volume of trade and asset returns, and Section 4 concludes. Appendix A gives necessary and sufficient conditions on a process which, if added to asset prices, will not distort the pricing kernels (and the one-period asset spans). Appendix B

[4]In their model there are no rational bubbles, as the value of a firm is always equal with its the present value of dividends, discounted at market rates. They use a different definition of bubbles, calculated as the difference between the value of the firm and the value predicted using the q theory of investment.

[5]In Kocherlakota (2009), non-entrepreneurs (called workers) are simply assumed to consume the fruits of their labor each period. The other three papers use overlapping generations models, in which the consumers are the old agents, who again simply consume their wealth. Farhi and Tirole (2012) allow also for the presence of an outside consumer sector endowed with one period Lucas trees, but again these agents just consume the present value of their Lucas tree endowment.

introduces some extensions to the example in Section 4.

2 A guiding example

We consider an extension of the example in Kocherlakota (2011), or alternatively, a deterministic version of the model in Guerrieri and Lorenzoni (2011), which we can solve analytically. There is no uncertainty and only one good. There are two agents $\{1, 2\}$ with identical utilities $E\sum_{t\geq 0}\beta^t(\ln c_t - n_t)$ over consumption (c) and labor (n), where $0 < \beta < 1$. Agent 1 is productive in odd periods $\{1, 3, \ldots\}$, while agent 2 is productive in even periods $\{0, 2, 4, \ldots\}$. Agent i can convert 1 unit of labor into Z_t^i units of consumption, where $Z_t^i = 1$ if i is productive at t, and 0 otherwise. At any period, the productive agent is referred to as the high-type (agent), and the unproductive agent is the low-type.

There is one long-lived asset paying dividends $d_t := \lambda\eta^t$ at period $t \geq 0$, where $0 \leq \eta < 1$ and $\lambda > 0$, and one period bonds (needed only for the case when $\eta = 0$). Agent i is endowed with θ_{-1}^i units of the asset. Additionally, at each period t, agent i has an endowment of goods e_t^i, where $e_t^i := y^H$ if i is high-type at t and equal to y^L otherwise. We assume $y^H \leq 1$, $y^L < \beta$ and $\eta \leq y^L/\beta$.

For a period t, let θ_{t-1}^i be agent's i beginning of period asset holdings, p_t the (ex-dividend) price of the long-lived asset, and q_t be the price of the bond. Thus the budget constraints of agent i are

$$c_t^i + p_t\theta_t^i + q_t b_t = e_t^i + Z_t^i n_t^i + (p_t + d_t)\theta_{t-1}^i + b_{t-1}, \forall t \geq 0.$$

Agent i is subject to debt constraints restricting their beginning of period wealth,

$$(p_t + d_t)\cdot\theta_{t-1}^i + b_{t-1} \geq -B,\ \forall t \geq 0,$$

and B is assumed to satisfy

$$0 < B \leq \frac{1 - y^L}{1 + \beta}. \tag{2.1}$$

Let $R_{t+1} := 1/q_t$ be the gross interest rate (return) from date t to $t + 1$. The absence of arbitrage between bonds and the long-lived asset ensures that $p_t = (p_{t+1} + d_{t+1})/R_{t+1}$.

We analyze two cases, differing in the dividend process and asset structure. The first case is $\eta = 0$ and $\theta_{-1}^1 = \theta_{-1}^2 = \frac{1}{2}$, thus the asset is fiat money in unit supply. This case was studied in Kocherlakota (2011), where fiat money is referred to as "land". The difference is that we use constraints on debt limiting the beginning (rather than the end) of period wealth, in order to be in line with the literature on endogenous debt limits in economies with limited enforcement (Alvarez and Jermann 2000), and that we also study the transition dynamics to the steady state, to be able to analyze completely the welfare effects of a credit crunch and bubble injection. Agents can also trade in a complete set of one-period Arrow securities, which in this deterministic economy amount to one period bonds.[6] Next we focus on the case $\eta > 0$ and $\theta^1_{-1} = \theta^2_{-1} = 0$, thus the asset is in zero supply. This asset dynamically completes the markets as it has a positive price, hence bonds are unnecessary. Therefore if $\eta > 0$ (hence $d_t > 0$ and $p_t > 0$), we assume that bonds are not available for trade ($b_t = 0$ for all t). In both cases, agent i's combined endowment at t (non-dividend income e^i_t plus dividends from initial holdings $\theta^i_{-1}d_t$) is the same, e^i_t. This preserves the structure of equilibrium (transfers between agents and interest rates).

Iterating in $p_t = (p_{t+1} + d_{t+1})/R_{t+1}$ gives

$$p_0 = \sum_{t>0} \prod_{s=1}^{t} R_s^{-1} d_t + \lim_{t\to\infty} \prod_{s=1}^{t} R_s^{-1} p_t.$$

The term $\lim_{t\to\infty} \prod_{s=1}^{t} R_s^{-1} p_t$ represents the asset price in excess of the present value of its future dividends, and is referred to as a bubble whenever it is non-zero. Notice that for the fiat money case (the zero dividend case), any positive price for money represents a bubble.

We analyze first equilibria without bubbles, where the price of the asset equals the discounted present value of its future dividends. The two cases (zero and non-zero dividend) lead to an identical equilibrium, except for portfolios. The transition to the steady state is complete after the initial period. Agents' consumption in the transition period (zero) is c^H_0 for the high-type, and c^L_0 for the low-type, labor supply of the high-type is n_0, and the interest rate is R_1. In steady state, the high-type (low-type) agent has consumption c^i_t equal to c^H (c^L), wealth level $(p_t + d_t)\theta^i_{t-1} + b^i_{t-1}$ equal to $-B$ (B), his labor supply n^i_t is n (0) and the interest rates are constant, R.

[6] Without bonds, autarchy would be the only feasible allocation without a bubble (with unvalued money).

At the proposed path, agents' first order conditions are

$$\frac{1}{c^H} = 1; \quad \frac{c^L}{c^H} = \beta R; \quad \frac{c^H}{c^L} \geq \beta R, \tag{2.2}$$

$$\frac{1}{c_0^H} = 1; \quad \frac{c^L}{c_0^H} = \beta R_1; \quad \frac{c^H}{c_0^L} \geq \beta R_1. \tag{2.3}$$

Budget constraints are

$$c^H + R^{-1}B = y^H - B + n, \quad c^L - R^{-1}B = y^L + B,$$
$$c_0^H + R^{-1}B = y^H + n_0, \quad c_0^L - R^{-1}B = y^L.$$

Thus R is determined as the unique solution in $\left(\frac{y^L}{\beta}, \frac{1}{\beta}\right]$ of $\beta R = y^L + B(1 + R^{-1})$ $(= c^L)$, or equivalently of

$$B = \frac{\beta R - y^L}{1 + R^{-1}}. \tag{2.4}$$

Indeed, the right hand side of (2.4) is strictly increasing in R, and equal to 0 for $R = y^L/\beta$ and to $\frac{1-y^L}{1+\beta}$ for $R = \frac{1}{\beta}$. Consumption, labor supply and interest rates during transition are obtained as function of R from the budget constraints and the first order conditions,

$$c^H = c_0^H = 1, \; R_1 = R, \; c^L = \beta R, \; c_0^L = y^L + R^{-1}B, \tag{2.5}$$

$$n = 1 + B + R^{-1}B - y^H, \; n_0 = 1 + R^{-1}B - y^H. \tag{2.6}$$

Asset prices and portfolios θ_t^H, b_t^H (of the high-type) and θ_t^L, b_t^L (of the low-type) are

$$p_t = \sum_{s>t} R^{-(s-t)} d_s = \frac{\lambda \eta^{t+1}}{R - \eta}, \forall t \geq 0, \tag{2.7}$$

$$\theta_t^H = B/(p_{t+1} + d_{t+1}), \; \theta_t^L = -B/(p_{t+1} + d_{t+1}); \; b_t^H = b_t^L = 0 \quad \text{if } \eta > 0, \tag{2.8}$$

$$\theta_t^H = \theta_t^L = \frac{1}{2}; \; b_t^H = B, \; b_t^L = -B \quad \text{if } \eta = 0. \tag{2.9}$$

To confirm that (2.4)-(2.8) describe indeed an equilibrium, it remains to verify that the first order conditions of low-types hold, that consumption and labor supply are positive, and that transversality and market clearing conditions hold. The inequalities in (2.2)-(2.3) hold if $c^H \geq c^L$, or equivalently, if $R \leq 1/\beta$, which is true, by (2.1) (and (2.4)). Consumption and labor supply are positive by (2.5) and (2.6), since $y^H \leq 1$ by assumption. The transversality conditions[7] and the market clearing conditions are satisfied:

$$\lim_{t\to\infty} \frac{\beta^t}{c_t^i}((p_t + d_t)\theta_{t-1}^i + b_{t-1}^i + B) = 0,$$
$$\theta_t^1 + \theta_t^2 = 0,\ c^H + c^L = y^H + y^L + n,\ c_0^H + c_0^L = y^H + y^L + n_0.$$

We fully described the equilibrium associated to some debt limits $B \in \left(0, \frac{1-y^L}{1+\beta}\right]$. Notice that consumption in steady state is increasing in R, and hence in B (by (2.4)) as $c^L = \beta R$ and $c^H = 1$. Similarly, consumption during transition, labor supply (both during transition and in steady state) and interest rates are all increasing functions of B (and of R, by (2.4)). Indeed, they are increasing functions of $R^{-1}B$, and therefore of B , since

$$R^{-1}B = \beta - \frac{\beta + y^L}{1 + R}. \tag{2.10}$$

Intuitively, a credit crunch (a decrease in B) lowers the interest rates, as all agents want to save more. Constrained borrowers have to reduce their indebtedness, while unconstrained ones increase their savings for precautionary reasons. The borrowing constrained unproductive agents adjust by consuming less as they cannot work more, while the productive agents reduce their labor supply due to the low return on saving. As shown computationally (for plausible parametrizations) in Guerrieri and Lorenzoni (2011), even when indebted agents can adjust by both spending less and working more, the consumption side dominates and output falls.

The welfare of the agents is also lower after a credit crunch, since the utilities of the first agent (initially low-type) and second agent (initially high-type) are

$$\begin{aligned} U^L &:= \ln c_0^L + \frac{\beta}{1-\beta^2}\left(\ln c^H - n + \beta \ln c^L\right) \\ &= \ln(y^L + R^{-1}B) + \frac{\beta^2}{1-\beta^2}(\ln R - R) + \frac{\beta}{1-\beta^2}(\beta \ln \beta + y^L + y^H - 1), \\ U^H &:= \ln c_0^H + \frac{\beta}{1-\beta^2}\left(\ln c^L - \beta n\right) = \frac{\beta}{1-\beta^2}\left(\ln R - \beta^2 R + \ln \beta + \beta y^L\right), \end{aligned}$$

[7]See Bidian and Bejan (2012) for their derivation.

and therefore are increasing functions of R (and B).[8]

Finally, trading volume decreases in the aftermath of a credit crunch. Indeed, in the case $\eta = 0$ (fiat money), by (3.9), the trading volume at t is $|q_t b_t^i| = R^{-1}B$, and therefore decreases when B decreases. For the case $\eta > 0$ (non-zero dividend asset), using (2.7) and (2.8), the trading volume at t is

$$|p_t(\theta_t^i - \theta_{t-1}^i)| = \begin{cases} R^{-1}B & \text{if} \quad t = 0 \\ R^{-1}B(1+\eta) & \text{if} \quad t > 0 \end{cases} . \tag{2.11}$$

Thus trading volume decreases when B (and R) decreases, by (2.10).

To illustrate some of the results of this paper, we focus on the equilibrium with zero net interest rates $R = \bar{R} := 1$, that is we assume that $B = \bar{B}$, where[9]

$$\bar{B} := \frac{\beta - y^L}{2}.$$

Barred variables will refer to this equilibrium.

We show that one can introduce a bubble of size $\varepsilon > 0$ in asset prices without perturbing the equilibrium allocations, as long as agent's i debt limits are tightened to $B - \theta_{-1}^i\varepsilon$, for each i. The idea is that asset prices higher by ε boost the initial wealth of agent's i by $\theta_{-1}^i\varepsilon$. The tighter future debt limits (reduced by $\theta_{-1}^i\varepsilon$) force the agent to save the additional wealth, sterilizing entirely the windfall created by the bubble.

Consider first the fiat money case, $\eta = 0$. Assume that a credit crunch occurs, and that the debt limits are tightened to $\hat{B} := \bar{B} - \varepsilon/2$, where $0 < \varepsilon < 2\bar{B}$. From (2.10), as discussed before, the credit crunch will contract consumption and labor supply. There exists however another "hatted" equilibrium under the tighter debt limits $\hat{B}$, equivalent from the point of view of consumption, labor supply and interest rates to the equilibrium before the credit crunch (the equilibrium associated to debt limits $\bar{B}$ and with unvalued

[8] Clearly $\frac{\partial U^H}{\partial R} > 0$ as $R \leq 1/\beta$. Similarly,

$$\frac{\partial U^L}{\partial R} = \frac{\beta + y^L}{y^L + R^{-1}B} \frac{1}{(1+R)^2} + \frac{\beta^2}{1-\beta^2}\left(\frac{1}{R} - 1\right),$$

which is a decreasing function of R. Moreover, $\left.\frac{\partial U^L}{\partial R}\right|_{R=1/\beta} = 0$, therefore $\frac{\partial U^L}{\partial R} > 0$ for $R < 1/\beta$.

[9] As known from Hellwig and Lorenzoni (2009) and Bidian and Bejan (2012), these are the endogenous debt limits that prevent default and allow for maximal credit expansion, when the penalty for default is an interdiction to borrow.

money),

$$\hat{c}^H = \bar{c}^H, \hat{c}^L = \bar{c}^L, \hat{c}_0^H = \bar{c}_0^H, \hat{c}_0^L = \bar{c}_0^L, \hat{n} = \bar{n}, \hat{n}_0 = \bar{n}_0, \hat{R} = \bar{R} = 1,$$

but with money valued at ε (thus with the long-lived asset' prices increased by ε),

$$\hat{p}_t = \bar{p}_t + \varepsilon = \varepsilon, \tag{2.12}$$

and with identical portfolio holdings. Indeed, it is immediate to check that agents' budget constraints are satisfied by the new prices and portfolios. The optimality and market clearing conditions remain the same. The equivalence of the two equilibria (the barred and the hatted one) is a particular instance of the "bubble equivalence theorem" to be analyzed in detail in the paper, for general stochastic economies, with possibly incomplete markets and debt limits endogenized as in Alvarez and Jermann (2000). It shows that a bubble can counteract entirely the effect of a credit crunch (the subsequent contraction), via the wealth injection to the asset holders. In this sense, bubbles are expansionary.

In the absence of a bubble, the equilibrium under the tighter debt limits $\hat{B}$ would lead to lower trading volume. Therefore it is not surprising that a bubble, by increasing the risk sharing to levels possible before the credit crunch, will also increase the trading volume. However, we ask the non-obvious and more interesting question throughout the paper: can a bubble lead to higher trading volume when compared to the equivalent (in terms of allocations) bubble-free equilibrium with more relaxed debt limits (before the credit crunch)? This would imply that bubbles can create additional trade beyond that needed to support the additional amount of risk-sharing enabled by them.

In the fiat money case studied above, it turns out that bubbles do not affect portfolios (when comparing the equivalent equilibria). The bubble-free (barred) equilibrium and the bubbly (hatted) equilibrium with tighter debt limits have identical allocations and portfolios of bonds and money. Therefore one could conclude that the bubble does not affect trading volumes. Of course, with redundant assets, quantifying trading volume effects of bubbles is an ill-posed problem. Due to the redundancy of either valued money or bonds, one can achieve arbitrarily desired trading volumes in the bubble-carrying asset (money), by appropriately adjusting the portfolios in the remaining assets (bonds).

Therefore throughout the paper we analyze the trading volume effects of bubbles only in environments without redundant assets, where the problem is well-posed and we can obtain an unambiguous answer. In the example of this section, we show that in case when $\eta > 0$

(hence the stock is the only asset), the bubble increases the trading volume even relative to the equivalent equilibrium with relaxed debt limits. We already know that without a bubble the credit crunch would result in lower trading volumes. We show below that, in fact, the bubble more than restores the trading volume to levels before the credit-crunch, and leads to net increases in trading volume. As in the fiat money case, it can be verified that the barred (bubble-free) equilibrium is equivalent with a hatted (bubbly) equilibrium, where the asset prices are higher by $\varepsilon > 0$ ($\hat{p}_t = p_t + \varepsilon$), the allocations are unchanged, and agents' portfolios are

$$\hat{\theta}^i_t = \frac{\bar{\theta}^i_t}{1+\Lambda_t}, \text{ where } \Lambda_t = \frac{\varepsilon}{\bar{p}_t}. \tag{2.13}$$

Using $\bar{\theta}^i_{t-1} = -\eta\bar{\theta}^i_t$ and $\Lambda_{t-1} = \eta\Lambda_t$, in the bubbly equilibrium the trading volume is

$$\begin{aligned} |\hat{p}_t(\hat{\theta}^i_t - \hat{\theta}^i_{t-1})| = \bar{p}_t(1+\Lambda_t)\left|\frac{\bar{\theta}^i_t}{1+\Lambda_t} - \frac{\bar{\theta}^i_{t-1}}{1+\Lambda_{t-1}}\right| &= |\bar{p}_t\bar{\theta}_t| \cdot \left(1 + \frac{\eta(1+\Lambda_t)}{1+\eta\Lambda_t}\right) \\ &= |\bar{p}_t(\bar{\theta}_t - \bar{\theta}_{t-1})| \cdot \frac{1}{1+\eta}\left(1 + \frac{\eta(1+\Lambda_t)}{1+\eta\Lambda_t}\right), \quad \forall t > 0. \end{aligned}$$

Thus the bubble injection strictly increases the trading volume for all periods $t > 0$, when compared to the bubble-free (barred) equilibrium, and the increase factor is:

$$1 < \frac{1}{1+\eta}\left(1 + \frac{\eta(1+\Lambda_t)}{1+\eta\Lambda_t}\right) = \frac{1}{1+\eta}\left(2 - \frac{1-\eta}{1+\eta\Lambda_t}\right) \nearrow \frac{2}{1+\eta} \text{ (as } t \to \infty).$$

The larger the ε is (that is, the initial value of the bubble), the closer is the initial relative increase in the trading volume to its asymptotic value $2/(1+\eta)$ (of relative increase). Notice that for small η, that is if we approach the case of the non-dividend paying asset (fiat money), the volume of trade doubles.

3 Bubble injections

We consider a stochastic, discrete-time, infinite horizon economy. The time periods are indexed by the set $\mathbb{N} := \{0, 1, \ldots\}$. The uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$ and by the filtration $(\mathcal{F}_t)_{t=0}^{\infty}$, which is an increasing sequence of finite partitions $\mathcal{F}_t$[10] on the set of states of the world Ω generating $\mathcal{F}$,[11] with $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

[10] We interpret $\mathcal{F}_t$ as the information available at period t.

[11] That is, such that $\mathcal{F} = \sigma(\cup_t \mathcal{F}_t)$, where $\sigma(\cup_t \mathcal{F}_t)$ represents the smallest σ-algebra containing $\cup_t \mathcal{F}_t$.

We let X be the set of all stochastic processes adapted to $(\mathcal{F}_t)_{t=0}^{\infty}$,[12] and denote by X_+ (respectively X_{++}) the processes $x \in X$ such that $x_t \geq 0$ P-almost surely (respectively $x_t > 0$ P-almost surely) for all $t \in \mathbb{N}$. All statements, equalities, and inequalities involving random variables are assumed to hold only "P-almost surely", and we will omit adding this qualifier. When $K, L \in \mathbb{N} \setminus \{0\}$, let $X^{K\times L}$, respectively $X_+^{K\times L}$ be the set of vector (or matrix) processes $(z^{ij})_{1\leq i\leq K, 1\leq j\leq L}$ with $z^{ij} \in X$, respectively $z^{ij} \in X_+$.

There is a single consumption good and a finite number, I, of consumers. An agent $i \in \{1, 2, \ldots, I\}$ has endowments $e^i \in X_+$, and his preferences are represented by a utility $U : X_+ \to \mathbb{R}$ given by $U^i(c) = E\sum_{t=0}^{\infty} u_t^i(c_t)$, where $u_t^i : \mathbb{R}_+ \to \mathbb{R}$ is continuous, increasing and concave and $E(\cdot)$ is the expectation operator with respect to probability P. The conditional expectation given the information available at t, $\mathcal{F}_t$, is denoted by $E_t(\cdot)$. Since there is no information at period 0, $E_0(\cdot) = E(\cdot)$. The continuation utility of agent i at t provided by a consumption stream $c \in X_+$ is $U_t^i(c) := E_t \sum_{s\geq t} u_s^i(c_s)$.

There is a finite number J of infinitely lived, disposable securities, traded at every date. The dividend and price vector processes are $d = (d^1, \ldots, d^J) \in X_+^{1\times J}$ and $p = (p^1, \ldots, p^J) \in X_+^{1\times J}$.

Consumer i has an initial endowment $\theta_{-1}^i \in \mathbb{R}_+^J$ of securities and his trading strategy is represented by a process $\theta^i \in X^{J\times 1}$. Fix some wealth bounds $w^i \in X$ for agent i and define the budget constraint and indirect utility of an agent i from period $s \geq 0$ onward, when faced with prices $p \in X_+^{1\times J}$, debt bounds $w^i \in X$ and having an initial wealth $\nu_s : \Omega \to \mathbb{R}$ which is $\mathcal{F}_s$-measurable, as

$$B_s^i(\nu_s, w^i, p) = \{(c^i, \theta^i) \in X_+ \times X^{J\times 1} \mid c_s^i + p_s\theta_s^i = e_s^i + \nu_s, \tag{3.1}$$

$$c_t^i + p_t\theta_t^i = e_t^i + (p_t + d_t)\theta_{t-1}^i, (p_t + d_t)\theta_{t-1}^i \geq w_t^i, \forall t > s\},$$

$$V_s^i(\nu_s, w^i, p) = \max_{(c^i,\theta^i)\in B_s^i(\nu_s,w^i,p)} U_s^i(c^i). \tag{3.2}$$

Definition 3.1. *A vector $\big(p, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\big)$ consisting of a security price process $p \in X_+^{1\times J}$, and for each agent $i \in \{1, \ldots, I\}$, a consumption process $c^i \in X_+$ and a trading strategy (portfolios) $\theta^i \in X^{J\times 1}$ is an equilibrium with exogenous debt limits $(w^i)_{i=1}^I$ if the following conditions are met:*

i. Consumption and portfolios of each agent i are feasible and optimal: $(c^i, \theta^i) \in B_0^i((p_0 + d_0)\theta_{-1}^i, w^i, p)$ and $U^i(c^i) = V_0^i\big((p_0 + d_0)\theta_{-1}^i, w^i, p\big)$.

[12] This is the set of sequences $x = (x_t)_{t\in\mathbb{N}}$ of random variables $x_t : \Omega \to \mathbb{R}$ such that x_t is $\mathcal{F}_t$-measurable.

ii. Markets clear: $\sum_{i=1}^{I} c_t^i = \sum_{i=1}^{I} e_t^i + d_t \cdot \sum_{i=1}^{I} \theta_{-1}^i, \quad \sum_{i=1}^{I} \theta_t^i = \sum_{i=1}^{I} \theta_{-1}^i,\ \forall t \in \mathbb{N}$.

Consider an equilibrium $\left(p, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ with (exogenous) debt bounds $(w^i)_{i=1}^I$. Since the utilities of the agents are strictly increasing in consumption at each date and state, prices p exclude arbitrage opportunities. Thus there cannot exist $\theta \in X^{J\times 1}$ and $t \in \mathbb{N}$ such that $p_t\theta_t \leq 0$ and $(p_{t+1} + d_{t+1})\theta_t \geq 0$, with at least one inequality being strict on a set of positive probability. Otherwise consumer i would alter his portfolio θ_t^i at t by adding to it the strategy θ_t, guaranteeing an increase in his consumption at t and $t+1$, and a strict increase in one of the periods, with positive probability. This modified strategy still satisfies the debt constraints. The absence of arbitrage opportunities is equivalent to the existence of a process $a \in X_{++}$ such that (Santos and Woodford 1997)

$$p_t = E_t\left[\frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1})\right], \forall t \geq 0. \tag{3.3}$$

We denote by $A(p)$ the set of all processes $a \in X$ satisfying equation (3.3), and we call them *deflators.* Strictly positive deflators belonging to $A_{++}(p) := A(p) \cap X_{++}$ will be called *state price densities*, or (interchangeably) *pricing kernels.* These pricing kernels following from the absence of arbitrage opportunities can be used to define the "fundamental value" of an asset. Equation (3.3) implies that $p_t = \frac{1}{a_t} E_t \sum_{s>t} a_s d_s + \lim_{T\to\infty} \frac{1}{a_t} E_t a_T p_T$, and

$$b_t(a, p) := \frac{1}{a_t} \lim_{T\to\infty} E_t a_T p_T \tag{3.4}$$

is well defined and nonnegative, and for all $t \in \mathbb{N}$,

$$a_t b_t(a, p) = E_t a_{t+1} b_{t+1}(a, p). \tag{3.5}$$

Therefore $a \cdot b(a, p)$ is a nonnegative martingale, and $b(a, p) = 0$ if and only if $b_0(a, p) = \frac{1}{a_0} \lim_{t\to\infty} E a_t p_t = 0$. We interpret the discounted present value of dividends d under the state price density a, that is $f_t(a) := \frac{1}{a_t} E_t \sum_{s>t} a_s d_s$, as the *fundamental value* of d. Hence $b(a, p)$ represents the part of asset prices in excess of fundamental values. Following Santos and Woodford (1997), we say that the equilibrium price process p *ambiguously involves a bubble* if $b_0(a, p) > 0$ for some $a \in A_{++}(p)$, while $b_0(a', p) = 0$ for some other $a' \in A_{++}(p)$. If $b_0(a, p) > 0$ for all $a \in A_{++}(p)$, the equilibrium prices *unambiguously involves a bubble* component.

Kocherlakota (2008) assumed that in addition to trading in long-lived securities, agents can also trade in each period a full set of state-contingent claims to consumption next period. Given an equilibrium without bubbles in which the asset prices are p and the state price density[13] is a, and given an arbitrary process $\varepsilon \in X_+^{1\times J}$ such that $a \cdot \varepsilon$ is a martingale, he showed that an "equivalent" equilibrium with prices $p+\varepsilon$, unchanged pricing kernel a and identical consumption paths for the agents can be constructed. He dubbed this result the "bubble equivalence theorem", since the process ε "injected" in the asset prices is the bubble component for the price process $p+\varepsilon$, that is $\varepsilon = b(a, p+\varepsilon)$.

We show that Kocherlakota's (2008) bubble equivalence theorem holds in our incomplete markets framework, if the candidate processes to be injected in asset prices are nonnegative processes that preserve the set of deflators. Formally, a process $\varepsilon \in X^{1\times J}$ is *deflator-preserving* (given prices p) if $A(p) = A(p+\varepsilon)$, and we denote the set of all such processes by

$$M^J(p) := \left\{\varepsilon \in X^{1\times J} \mid A(p) = A(p+\varepsilon)\right\}. \tag{3.6}$$

Let $M_+^J(p) := M^J(p) \cap X_+^{1\times J}$. We give several equivalent characterization of the set $M^J(p)$ in Appendix A. It is also the set of processes ε that are a martingale when discounted by any deflator in $A(p)$ and $A(p+\varepsilon)$. Equivalently, $M^J(p)$ is also the set of all discounted martingales (under some $a \in A(p)$) that preserve the asset span if added to asset prices. Finally, $M^J(p)$ is the set of processes ε with rates of growth replicable by returns on portfolios under the initial prices p and the adjusted prices $p+\varepsilon$ (if the process is added to asset prices). Thus there are some portfolios $\Lambda = (\Lambda^1, \ldots, \Lambda^J) \in X^{J\times J}$ and $\Gamma = (\Gamma^1, \ldots, \Gamma^J) \in X^{J\times J}$ such that

$$\frac{\varepsilon_{t+1}^j}{\varepsilon_t^j} = \frac{(p_{t+1}+d_{t+1})\Lambda_t^j}{p_t \Lambda_t^j} = \frac{(\hat{p}_{t+1}+d_{t+1})\Gamma_t^j}{\hat{p}_t \Gamma_t^j}, \forall t \geq 0, \forall j \in \{1, \ldots, J\},$$

where $\hat{p} = p + \varepsilon$.

As shown in Appendix A, the set $M^J(p)$ is large, as it contains the set $\bar{M}^J(p)$ (and is equal to it if there are no redundant assets) defined by

$$\bar{M}^J(p) := \Big\{\varepsilon \in X^{1\times J} \mid \exists \Lambda = (\Lambda^1, \ldots, \Lambda^J) \in X^{J\times J} \text{ s.t. } \forall t \geq 0, j \in \{1, \ldots, J\}, \\ \det(\mathbf{I} + \Lambda_t) \neq 0 \text{ and } \varepsilon_0^j = p_0 \Lambda_0^j, \quad \frac{\varepsilon_{t+1}^j}{\varepsilon_t^j} = \frac{(p_{t+1}+d_{t+1})\Lambda_t^j}{p_t \Lambda_t^j}\Big\}, \tag{3.7}$$

[13]The pricing kernel is unique when markets are complete .

where $\mathbf{I}$ denotes the J-dimensional identity matrix, and $\det(\cdot)$ is the determinant of a matrix. Therefore any trading strategy $\Lambda = (\Lambda^1, \ldots, \Lambda^J) \in X^{J\times J}$ with $\det(\mathbf{I} + \Lambda_t) \neq 0$ can be used to construct an $\varepsilon \in \bar{M}^J(p) \subset M^J(p)$. Moreover, such a Λ with nonnegative *gross* returns (for example, if $\Lambda \in X_+^{J\times J}$) generates a nonnegative ε ($\varepsilon \in M_+^J(p)$). The condition $\det(\mathbf{I} + \Lambda_t) \neq 0$ guarantees that there is no drop of rank in the asset span after ε is injected in asset prices. Non-singularity of $\mathbf{I} + \Lambda_t$ is a mild condition (generically satisfied), equivalent with requiring that -1 is not an eigenvalue of Λ_t. In particular, when a bubble is injected only in one of the assets, say the first, then $\Lambda_t = (\Lambda_t^1, 0, \ldots, 0)$, and $\det(\mathbf{I} + \Lambda_t) \neq 0$ whenever $\Lambda_t^{11} \neq -1$, as $(\mathbf{I} + \Lambda_t)^{-1} = \mathbf{I} - \Lambda_t/(1 + \Lambda_t^{11})$. For $\varepsilon \in \bar{M}^J(p)$, the set of trading strategies with returns replicating the growth in ε is denoted by

$$\Lambda(\varepsilon, p) := \left\{\Lambda \in X^{J\times J} \mid \forall t \geq 0,\ \varepsilon_t = p_t\Lambda_t,\ \varepsilon_{t+1} = (p_{t+1} + d_{t+1})\Lambda_t,\ \det(\mathbf{I} + \Lambda_t) \neq 0\right\}.$$

If there are no redundant assets, the set $\Lambda(\varepsilon, p)$ is a singleton, and $\bar{M}^J(p) = M^J(p)$ (Proposition A.2).

A bubble-free equilibrium with prices p and a bubbly equilibrium with prices $p + \varepsilon$ cannot be equivalent *unless* $\varepsilon \in M_+^J(p)$, otherwise the set of pricing kernels would differ, $A(p) \neq A(p + \varepsilon)$. We will show also the converse, that *any* $\varepsilon \in M_+^J(p)$ can be injected as a bubble in prices, resulting in an equivalent equilibrium from the point of view of pricing kernels and agents' consumption. We prove first that agents' feasible consumption paths remain unchanged when prices are inflated by a process in $M^J(p)$, if the debt limits are tightened appropriately.

Proposition 3.1. *Consider an agent i starting period t with wealth equal to ν_t, assumed $\mathcal{F}_t$-measurable. Let $\bar{\theta}_{-1} : \Omega \to \mathbb{R}^J$ be $\mathcal{F}_t$-measurable and $\varepsilon \in M^J(p)$. Then*

$$(c^i, \theta^i) \in B_t^i\left(\nu_t, w^i, p\right) \Longleftrightarrow (c^i, \hat{\theta}^i) \in B_t^i\left(\hat{\nu}_t, \hat{w}^i, \hat{p}\right),$$

where $\hat{\nu}_t = \nu_t + \varepsilon_t\bar{\theta}_{-1}$, $\hat{w}_t^i = w^i + \varepsilon\bar{\theta}_{-1}$, $\hat{p} = p + \varepsilon$, and $\theta^i, \hat{\theta}^i$ satisfy $(p_s + d_s)(\theta_{s-1}^i - \bar{\theta}_{-1}) = (\hat{p}_s + d_s)(\hat{\theta}_{s-1}^i - \bar{\theta}_{-1})$, for all $s > t$. Moreover, if $\varepsilon \in \bar{M}^J(p)$, then $\hat{\theta}_s^i = (\mathbf{I} + \Lambda_s)^{-1}\left(\theta_s^i + \Lambda_s\bar{\theta}_{-1}\right)$, where $\Lambda \in \Lambda(\varepsilon, p)$.

Proof. Let $(c^i, \theta^i) \in B_t^i\left(\nu_t, w^i, p\right)$. Since $\varepsilon \in M^J(p)$, there exists $\hat{\theta}^i$ such that $(p_s + d_s)(\theta_{s-1}^i - \bar{\theta}_{-1}) = (\hat{p}_s + d_s)(\hat{\theta}_{s-1}^i - \bar{\theta}_{-1})$, for all $s > t$. As $a \cdot \varepsilon$ is a martingale for each

$a \in A(p) = A(p+\varepsilon)$,

$$\hat{p}_s(\hat{\theta}^i_s - \bar{\theta}_{-1}) = E_s \frac{a_{s+1}}{a_s}(p_{s+1} + d_{s+1})(\theta^i_s - \bar{\theta}_{-1}) = p_s(\theta^i_s - \bar{\theta}_{-1}), \quad \forall s \geq t.$$

It follows that for $s \geq t+1$,

$$(\hat{p}_s + d_s)\hat{\theta}^i_{s-1} - \hat{p}_s\hat{\theta}^i_s = (p_s + d_s)\theta^i_{s-1} - p_s\theta^i_s.$$

Moreover,

$$\hat{\nu}_t - \hat{p}_t\hat{\theta}^i_t = \nu_t + \varepsilon_t\bar{\theta}_{-1} - (p_t + \varepsilon_t)\hat{\theta}^i_t = \nu_t - p_t\theta^i_t.$$

The debt limits are satisfied, as

$$(\hat{p}_t + d_t)\hat{\theta}^i_{t-1} = (p_t + d_t)\theta^i_{t-1} + \varepsilon_t\bar{\theta}_{-1} \geq w^i_t + \varepsilon_t\bar{\theta}_{-1} = \hat{w}^i_t,$$

and we conclude that $B^i_t\left(\nu_t, w^i, p\right) \subset B^i_t\left(\hat{\nu}_t, \hat{w}^i, \hat{p}\right)$. If $\varepsilon \in \bar{M}^J(p)$, then for $\Lambda \in \Lambda(\varepsilon, p)$

$$\begin{aligned}(p_s + d_s)(\theta^i_{s-1} - \bar{\theta}_{-1}) &= (p_s + d_s)(\mathbf{I} + \Lambda_{t-1})(\mathbf{I} + \Lambda_{t-1})^{-1}(\hat{\theta}^i_{s-1} - \bar{\theta}_{-1}) \\ &= (\hat{p}_s + d_s)(\mathbf{I} + \Lambda_{t-1})^{-1}(\theta^i_{s-1} - \bar{\theta}_{-1}),\end{aligned}$$

and therefore we could choose $\hat{\theta}^i_{s-1} - \bar{\theta}_{-1} := (\mathbf{I} + \Lambda_{t-1})^{-1}(\theta^i_{s-1} - \bar{\theta}_{-1})$, or equivalently, $\hat{\theta}^i_s = (\mathbf{I} + \Lambda_s)^{-1}\left(\theta^i_s + \Lambda_s\bar{\theta}_{-1}\right)$.

Conversely, using $-\varepsilon \in M^J(p)$ and the previously shown implication,

$$B^i_t(\hat{\nu}_t, \hat{w}^i, \hat{p}) \subset B^i_t(\hat{\nu}_t - \varepsilon_t\bar{\theta}_{-1}, \hat{w}^i - \varepsilon\bar{\theta}_{-1}, \hat{p} - \varepsilon) = B^i_t(\nu_t, w^i, p).$$

□

The intuition for the proposition is especially transparent in the particular case when $\bar{\theta}_{-1}$ is the portfolio with which the agent starts period t and ν_t is the value of this portfolio , $\nu_t := (p_t + d_t)\bar{\theta}_{-1}$.[14] With bubble-inflated prices, the owners of the assets receive a windfall in the form of higher initial wealth. Tightening their future debt bounds by the bubble weighted by initial asset holdings will force them to save the initial windfall in order to meet the more stringent borrowing requirements, leading thus to equivalent budget constraints.

[14] The slightly more general statement of Proposition 3.1 is needed in Theorem 3.3 and the discussion that follows after that theorem.

Consider a bubble-free equilibrium with asset prices p. For any process $\varepsilon \in M_+^J(p)$, we show that there is an equivalent equilibrium with prices $p+\varepsilon$, identical consumption and state price densities, and in which the debt constraints bind in exactly the same date and states (even though they differ). Moreover ε is the bubble component in the prices $p+\varepsilon$ for any state price density $a \in A(p+\varepsilon)(=A(p))$, that is $\varepsilon = b(a, p+\varepsilon)$, hence the new equilibrium unambiguously involves a bubble. The converse is also true. Consider an equilibrium that has a bubble under some deflator. If the set of deflators at fundamental and bubbly prices coincide (which guarantees, by Lemma A.1 that the bubble is unambiguous), then one can construct an equivalent bubble-free equilibrium (from the point of view of consumption and pricing kernels), but with relaxed debt limits, proportional to the size of the bubble.

Theorem 3.2. *Let* $\mathcal{E} := \left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ *be an equilibrium (with exogenous debt limits) and without bubbles. Choose* $\varepsilon \in M_+^J(p)$ *and* $\Lambda \in \Lambda(\varepsilon, p)$. *Then* $\hat{\mathcal{E}} := \left(\hat{p}, (\hat{w}^i)_{i=1}^I, (c^i, \hat{\theta}^i)_{i=1}^I\right)$ *is an equilibrium with (unambiguous) bubble* ε, *where*

$$\hat{p} = p + \varepsilon,\ \hat{w}^i = w^i + \varepsilon\theta_{-1}^i,\ (\hat{p}_t + d_t)(\hat{\theta}_{t-1}^i - \theta_{-1}^i) = (p_t + d_t)(\theta_{t-1}^i - \theta_{-1}^i), \forall t \geq 1. \quad (3.8)$$

Moreover, if $\varepsilon \in \bar{M}^J(p)$, *then* $\hat{\theta}_t^i = (\mathbf{I} + \Lambda_t)^{-1}\left(\theta_t^i + \Lambda_t\bar{\theta}_{-1}\right)$, *where* $\Lambda \in \Lambda(\varepsilon, p)$. *Conversely, assume that* $\mathcal{E}$ *has a bubble component, that is, there exists* $a \in A_{++}(p)$ *such that* $\varepsilon := b(a,p) \neq 0$. *If* $-\varepsilon \in M^J(p)$, *then* $\hat{\mathcal{E}}$ *given by* (3.8) *in which* ε *is replaced throughout by* $-\varepsilon$ *is a bubble-free equilibrium. Moreover, if* $-\varepsilon \in \bar{M}^J(p)$, *then* $\hat{\theta}_t^i = (\mathbf{I} + \Lambda_t)^{-1}\left(\theta_t^i + \Lambda_t\bar{\theta}_{-1}\right)$, *where* $\Lambda \in \Lambda(-\varepsilon, p)$.

Proof. For $i \in \{1, \ldots, I-1\}$, construct $\hat{\theta}^i$ satisfying (3.8), and set $\hat{\theta}^I := -\sum_{i=1}^{I-1}\hat{\theta}^i$ (which therefore also satisfies (3.8)). By construction, the market clearing $\sum_i \hat{\theta}^i = \sum_i \theta_{-1}^i$ holds. For each $i \in \{1, \ldots, I\}$, optimality of $(c^i, \hat{\theta}^i)$ in the set $B_0^i((\hat{p}_0 + d_0)\theta_{-1}^i, \hat{w}, \hat{p})$ follows from the optimality of (c^i, θ^i) in $B_0^i((p_0 + d_0)\theta_{-1}^i, w, p)$, and the equality of these two budgets (Proposition 3.1).

If $\varepsilon \in \bar{M}^J(p)$, optimality of $(c^i, \hat{\theta}^i)$ in the set $B_0^i((\hat{p}_0 + d_0)\theta_{-1}^i, \hat{w}, \hat{p})$ follows again from Proposition 3.1. Notice that $\sum_i \hat{\theta}_t^i = (\mathbf{I} + \Lambda_t)^{-1}(\sum_i \theta_t^i + \Lambda\sum_i \theta_{-1}^i) = \sum_i \theta_{-1}^i$, since $\sum_i \theta_t^i = \sum_i \theta_{-1}^i$. Thus the market clearing conditions are satisfied.

The converse can be established in an identical manner. □

The "bubble equivalence" result in Theorem 3.2 shows that any bubble that preserves the set of deflators (or the asset span) is effectively equivalent with an appropriate relaxation of debt limits, proportional with the size of the bubble.[15] If markets are complete at prices p (in the equilibrium $\mathcal{E}$), then the set $A(p)$ is a singleton, $A(p) = \{a\}$. A nonnegative process ε such that $a \cdot \varepsilon$ is a martingale does not necessarily lead to an equivalent equilibrium (with tighter debt limits) when added to prices p, since markets could become incomplete. The exact condition that guarantees that markets remain complete at the inflated prices is $\varepsilon \in M^J(p)$. However, if a full set of one-period Arrow securities is traded at each date and state, in addition to the long-lived securities, than there cannot be a drop-in-rank (a decrease in the asset span) when ε is added to prices p. Therefore any nonnegative process that is a martingale when discounted by the (unique) pricing kernel can be injected in asset prices and lead to an equivalent equilibrium with tighter debt limits. This is the case treated in Kocherlakota (2008).

We allow next for the *endogenous* determination of debt constraints driven by limited commitment/imperfect enforcement as in Alvarez and Jermann (2000), and show that the bubble inflated debt bounds in the equivalent bubbly equilibrium are also compatible with the endogenous mechanism determining debt limits.

Assume that at any period t, when facing prices p (and dividends d), consumer i can choose to default on his beginning of period debt[16] and leave the economy, receiving a continuation utility after default $\widetilde{V}_t^i(p)$ ($\mathcal{F}_t$-measurable). We allow this continuation utility to depend on exogenous variables such as endowments and dividends, but we make explicit only the functional dependence on prices, which are endogenous. Thus the *default penalty* for each agent i is described by a mapping $\widetilde{V}^i : X_+^{1\times J} \to X$. Alvarez and Jermann (2000), following Kehoe and Levine (1993), worked under the assumption that agents are banned from trading following default, hence for each agent i,

$$\widetilde{V}_t^i(p) := U_t^i(e^i). \tag{3.9}$$

Alternatively, Hellwig and Lorenzoni (2009), building on the work of Bulow and Rogoff (1989), assume that agents are subject to a milder punishment than (3.9). Agents can

[15] Proposition 3.1 and Theorem 3.2 are valid, without changes, if agents are subject to borrowing constraints of the form $p_t\theta_t^i \geq w_t^i$ rather than $(p_t + d_t)\theta_{t-1}^i \geq w_t^i$ as in (3.1). Indeed, for any $A \in A(p) = A(\hat{p})$ (and with the notation in (3.8)),

$$\hat{p}_t(\hat{\theta}_t^i - \theta_{-1}^i) = E_t\frac{a_{t+1}}{a_t}(\hat{p}_{t+1} + d_{t+1})(\hat{\theta}_t^i - \theta_{-1}^i) = E_t\frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1})(\theta_t^i - \theta_{-1}^i) = p_t(\theta_t^i - \theta_{t-1}^i),$$

hence $\hat{p}_t\hat{\theta}_t^i = p_t\theta_t^i + \varepsilon_t\theta_{-1}^i \geq w_t^i + \varepsilon_t\theta_{-1}^i$.

[16] This is equal to $(p_t + d_t)\theta_{t-1}^i$ if his trading strategy is $\theta^i \in X^{J\times 1}$.

continue to lend but not to borrow following default,

$$\widetilde{V}_t^i(p) := V_t^i(0, 0, p), \tag{3.10}$$

where the second argument in $V_t^i(0, 0, p)$ is the process in X identically equal to zero. As in Alvarez and Jermann (2000), the option to default endogenizes the debt limits to the maximum level so that repayment is always individually rational given future debt limits. This leads to the notion of debt limits that are *not-too-tight*.

Definition 3.2. *Debt limits w^i faced by agent i are not-too-tight (NTT) given prices p and penalties $\widetilde{V}^i : X_+^{1\times J} \to X$ if $V_t^i(w_t^i, w^i, p) = \widetilde{V}_t^i(p), \forall t$.*

The definition captures the idea that the bounds w^i have to be "tight enough" to prevent default, that is to be "self-enforcing" ($V_t^i(w_t^i, w^i, p) \geq \widetilde{V}_t^i(p)$), but they should allow for maximum credit expansion (thus one should not have $V_t^i(w_t^i, w^i, p) > \widetilde{V}_t^i(p)$ on a positive probability set). One can envision the NTT debt limits as being set by competitive financial intermediaries, with agents unable to trade directly with each other. The intermediaries set debt limits such that default is prevented, but credit is not restricted unnecessarily, since competing intermediaries could relax them and increase their profits.

We extend our definition of equilibrium to allow for the endogenous determination of debt constraints, in the presence of an outside option to default. An *Alvarez-Jermann equilibrium* (*AJ*-equilibrium, for short) $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ consists of a security price process $p \in X_+^{1\times J}$, and for each agent $i \in \{1, \ldots, I\}$, debt limits $w^i \in X$, a consumption process $c^i \in X_+$, a trading strategy $\theta^i \in X^{J\times 1}$ and a mapping $\widetilde{V}^i$ from prices and dividends into continuation utilities after default such that $\left(p, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ is an equilibrium with (exogenous) debt limits $(w^i)_{i=1}^I$, and w^i are not-too-tight given penalties for default $\widetilde{V}^i(p)$.[17]

Existence of *AJ*-equilibria in this general environment is a delicate problem, due to the presence of incomplete markets, real (long-lived) securities and infinite horizon, which creates existence problems even for equilibria with exogenous debt limits as in definition

[17] An equilibrium $\left(p, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ with exogenous debt limits $(w^i)_{i=1}^I$ can be transformed into an *AJ*-equilibrium in a rather trivial (and uninteresting) way by appropriately choosing the penalties for default. Indeed, $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ with $\widetilde{V}_t^i(p) := V_t^i(w_t^i, w^i, p)$ is trivially an *AJ*-equilibrium (as long as the indirect utility $V_t^i(w_t^i, w^i, p)$ is well defined). Alternatively, the initial equilibrium with exogenous debt bounds $(w^i)_{i=1}^I$ is also an *AJ*-equilibrium with new debt bounds $\bar{w}_t^i := (p_t + d_t)\theta_{t-1}^i$ and penalties $\widetilde{V}_t^i := U_t^i(c^i)$.

3.1. When markets are complete and the punishment for default is given by (3.9), the existence of the AJ-equilibrium is established by Kehoe and Levine (1993) and Alvarez and Jermann (2000). With incomplete markets, Hernandez and Santos (1996) show that in our environment, an equilibrium with exogenous debt limits exists for a dense subset of endowment and dividend processes, if agents are impatient, have a nonnegative initial holding of securities, and if their debt is restricted by the present value of future endowments,

$$w_t^i = - \inf_{a \in A_{++}(p)} E_t \sum_{s \geq t} \frac{a_s}{a_t} e_s^i. \tag{3.11}$$

The debt limits in (3.11) are chosen equal to the maximum amount that an agent can borrow, if he must hold nonnegative wealth after some finite date. With complete markets, they are the NTT debt limits when the punishment for default is the confiscation of endowment.

We show next that a bubble injection as in Theorem 3.2 preserves the NTT property of the debt limits, leading to the bubble equivalence theorem for AJ-equilibria:

Theorem 3.3. *Let* $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ *be an AJ-equilibrium. Choose* $\varepsilon \in M_+^J(p)$ *and* $\Lambda \in \Lambda(\varepsilon, p)$*. If* $\widetilde{V}^i(p+\varepsilon) = \widetilde{V}^i(p)$ *for all agents* $i \in \{1, \ldots, I\}$*, then* $\left(\hat{p}, (\hat{w}^i)_{i=1}^I, (c^i, \hat{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ *is an AJ-equilibrium, with* $\hat{p}, \hat{\theta}^i, \hat{w}^i$ *given by* (3.8).

Proof. By Proposition 3.1, $\hat{w}^i := w^i + \varepsilon \theta_{-1}^i$ are not-too-tight for prices $\hat{p}$, since

$$\widetilde{V}_t^i(\hat{p}) = \widetilde{V}_t^i(p) = V_t^i(w_t^i, w^i, p) = V_t^i(w_t^i + \varepsilon_t \theta_{-1}^i, w^i + \varepsilon \theta_{-1}^i, p + \varepsilon) = V_t^i(\hat{w}_t^i, \hat{w}^i, \hat{p}).$$

The conclusion follows from Theorem 3.2. □

The requirement $\widetilde{V}^i(p+\varepsilon) = \widetilde{V}^i(p)$ that the "penalty" continuation utilities after default are not affected by a bubble injection is necessary and sufficient to ensure that the equivalence result in Theorem 3.2 extends to AJ-equilibria. Indeed, agents continuation utilities when starting with maximal (binding) amounts of debt are identical in the bubbly and bubble-free equilibrium, and therefore the penalties for default with and without a bubble have to coincide, by the definition of NTT debt limits. This condition holds when the continuation utilities after default are of the form (3.9), or more generally when $\widetilde{V}^i$ does not depend on prices. These are the only types of penalties considered in Kocherlakota (2008). It holds also for the interdiction to borrow after default (3.10). In fact, it holds for a much more general class of penalties, where after default, an agent i is subjected to some exogenous debt limits $\widetilde{w}^i$ (equal to zero for an interdiction to borrow). Indeed, using Proposition 3.1 with $\nu_t := \widetilde{w}^i_t$, $\bar{\theta}_{-1} := 0 \in \mathbb{R}^J$ and $w^i := \widetilde{w}^i$, it follows that $V^i_t(\widetilde{w}^i_t + \varepsilon_t \cdot 0, \widetilde{w}^i, p + \varepsilon) = V^i_t(\widetilde{w}^i_t, \widetilde{w}^i, p)$, and therefore

$$\widetilde{V}^i_t(p + \varepsilon) = V^i_t(\widetilde{w}^i_t, \widetilde{w}^i, p + \varepsilon) = V^i_t(\widetilde{w}^i_t, \widetilde{w}^i, p) = \widetilde{V}^i_t(p). \tag{3.12}$$

Santos and Woodford (1997) show that when agents' debt limits are nonpositive, bubbles in assets in positive supply can exist only if the interest rates are low, leading to an infinite present value of consumption. Therefore the modified debt limits $(\hat{w}^i)$ of the bubbly equilibrium of Theorem 3.3 can remain nonpositive (assuming that the initial constraints (w^i) were nonpositive) only if interest rates are low. Low interest rates arise naturally under enforcement limitations, in order to induce agents to repay their debt and prevent default. Bidian and Bejan (2012) give examples of bubble injections leading to nonpositive debt limits when the penalties for default are the interdiction to trade (3.9), the interdiction to borrow (3.10), or a temporary interdiction to trade.

As applications of Theorems 3.2-3.3, we investigate next the effects of bubble injections on trading volume and asset returns.

4 Effects of bubbles on trading volume

Bubbles in an asset are typically associated with large increases in trading volume in that asset (Cochrane 2002). As seen in the example of Section 2, bubbles restore the equilibrium allocations to levels possible under relaxed debt limits (before the credit crunch). The relaxation in debt limits and the additional risk sharing enabled by a bubble leads (not surprisingly) to higher trading. The unexpected finding there was that a bubble increases the trading volume even when compared to the equivalent bubble-free equilibrium with relaxed debt limits (without a credit crunch). Thus bubbles create trading volume in excess of the "fundamental" trading volume needed to sustain the same level of risk sharing in the absence of the bubble.

We investigate the effect of bubble injections on the trading volume in more general environments. We compare the two "equivalent" equilibria of Theorem 3.2, the bubble-free equilibrium $\left(p, (w^i)^I_{i=1}, (c^i)^I_{i=1}, (\theta^i)^I_{i=1}\right)$ and the bubbly equilibrium $\left(\hat{p}, (\hat{w}^i)^I_{i=1}, (c^i, \hat{\theta}^i)^I_{i=1}\right)$. We assume that there are no redundant assets, which is needed to have a well-posed problem and a clearcut answer, as explained in Section 2. Without redundant assets, the set of deflator-preserving processes $M^J(p)$ equals $\bar{M}^J(p)$, the set of processes ε with growth that can be replicated by a non-singular trading strategy $\Lambda \in \Lambda(\varepsilon, p)$. Moreover, $\Lambda(\varepsilon, p)$ is a singleton, and therefore agents' adjustments in trading strategies in response to a bubble are unambiguous.

In the bubble-free equilibrium, the number of shares of each asset bought, respectively sold, by agent i at t is $(\theta_t^i - \theta_{t-1}^i)^+$, respectively $(\theta_t^i - \theta_{t-1}^i)^-$ (the positive part and the negative part of the change in portfolio are applied component-wise). Notice that the total number of shares of each asset bought and sold at t are equal, since

$$\sum_i (\theta_t^i - \theta_{t-1}^i)^+ = \sum_i (\theta_t^i - \theta_{t-1}^i)^- = \frac{1}{2} \sum_i |\theta_t^i - \theta_{t-1}^i|.$$

Thus we can measure the *share volume of trade* at t in each asset as

$$SV_t = (SV_t^1, \ldots, SV_t^J)' := \frac{1}{2} \sum_i |\theta_t^i - \theta_{t-1}^i|, \tag{4.1}$$

and the *dollar volume of trade* for asset j as $DV_t^j := p_t^j SV_t^j$. The share and dollar volume of trade in the bubbly equilibrium are $\hat{SV}_t := \frac{1}{2}\sum_i |\hat{\theta}_t^i - \hat{\theta}_{t-1}^i|$, $\hat{DV}_t^j := (p_t^j + \varepsilon_t^j)\hat{SV}_t^j, \forall j$, where $\hat{\theta}_t^i = (\mathbf{I} + \Lambda_t)^{-1}(\theta_t^i + \Lambda_t \theta_{-1}^i)$, for all $t \geq 0$ and $\Lambda \in \Lambda(\varepsilon, p)$.

In dynamically complete markets economies with Markov endowments and borrowing constraints that do not bind, Judd, Kubler, and Schmedders (2003) show that, with long-lived assets (stocks), there is no trade in equilibrium (generically in dividends), after a portfolio rebalancing that takes place in the first period, that is, $\theta_t^i = \theta_{t-1}^i$, for all i and $t \geq 1$. Notice that the trading volume of agent i after the bubble injection is

$$|\hat{\theta}_t^i - \hat{\theta}_{t-1}^i| = |\left((\mathbf{I} + \Lambda_t)^{-1} - (\mathbf{I} + \Lambda_{t-1})^{-1}\right)(\theta_t^i - \theta_{-1}^i)|.$$

Therefore, in this class of economies, a bubble ε injected in prices unambiguously increases the trading volume by distorting agents' portfolios, as long as the trading strategy $\Lambda = \Lambda(\varepsilon, p)$ replicating the rate of growth of the bubble displays time variation.

In fact a bubble in only one asset can create trading in all of the assets, as agents adjust their portfolios in response to the bubble-inflated price of that asset. To see this, assume, for concreteness, that the bubble develops in the first asset. Thus $\varepsilon := (\varepsilon^1, 0, \ldots, 0) \in X_+^{1 \times J}$.

In this case, $\Lambda = (\Lambda^1, 0, \ldots, 0) \in \Lambda(\varepsilon, p) \subset X^{J \times J}$, and

$$\hat{\theta}_t^i = \left(\mathbf{I} - \frac{\Lambda_t}{1+\Lambda_t^{11}}\right)(\theta_t^i + \Lambda_t \theta_{-1}^i) = \left(\mathbf{I} - \frac{\Lambda_t}{1+\Lambda_t^{11}}\right)\theta_t^i + \frac{\Lambda_t}{1+\Lambda_t^{11}}\theta_{-1}^i. \tag{4.2}$$

Therefore

$$|\hat{\theta}_t^i - \hat{\theta}_{t-1}^i| = \left(\left|\left(\frac{\Lambda_{t-1}^{11}}{1+\Lambda_{t-1}^{11}} - \frac{\Lambda_t^{11}}{1+\Lambda_t^{11}}\right)\theta_t^i\right|, \ldots, \left|\left(\frac{\Lambda_{t-1}^{J1}}{1+\Lambda_{t-1}^{11}} - \frac{\Lambda_t^{J1}}{1+\Lambda_t^{11}}\right)\theta_t^i\right|\right)',$$

and if Λ displays sufficient variation over time, then the trading volume in all assets increases.

The example in Appendix C illustrates the above discussion. It shows that the absence of trade in a bubble-free equilibrium can occur even with incomplete markets and debt limits not covered by Judd, Kubler, and Schmedders (2003), as long as the equilibrium is Pareto optimal. Trade is absent in the bubble-free equilibrium, and a bubble increases the trading volume. The share volume of trade vanishes asymptotically, but the effect on the dollar volume of trade is persistent and bounded away from zero.

Outside the class of frictionless economies studied by Judd, Kubler, and Schmedders (2003), it is difficult to analyze at our level of generality the effect of bubble injections on trading volume.

We focus therefore on a tractable two-agents model with complete markets and the interdiction to trade (3.9) as penalty for default, used by Alvarez and Jermann (2001) to make a compelling empirical case for the improved asset pricing performance of models with limited enforcement. In this model there is trade in equilibrium, but nevertheless bubbles (deterministic or stochastic) increase the trading volume. Small initial bubbles can generate very large increases (respectively drops) in the volume of trade while they run (respectively when they crash). There are also "contagion" effects, as a bubble in one asset boosts the trading volume also in the other asset.

Since there are only two agents, the portfolio of one agent determines fully the volume of trade in each security. In other words, for each agent $i \in \{1, 2\}$ and security j, $SV_t^j = |\theta_t^{i,j} - \theta_{t-1}^{i,j}|$ and $\hat{SV}_t^j = |\hat{\theta}_t^{i,j} - \hat{\theta}_{t-1}^{i,j}|$. Furthermore, there will be only two assets, and (4.2) becomes

$$\hat{\theta}_t^{i,1} = \frac{\theta_t^{i,1} - \theta_{-1}^{i,1}}{1+\Lambda_t^{11}} + \theta_{-1}^{i,1}; \quad \hat{\theta}_t^{i,2} = -\frac{\Lambda_t^{21}(\theta_t^{i,1} - \theta_{-1}^{i,2})}{1+\Lambda_t^{11}} + \theta_t^{i,2}. \tag{4.3}$$

The uncertainty is described by a time homogeneous Markov process $(s_t)_{t\in\mathbb{N}}$ with states $s_t \in \{1, 2\}$, and with a probability of reversal equal to $\pi \in (0, 1]$. Thus for any t, $s_{t+1} \neq s_t$ with probability π. The case $\pi = 1$ generates a deterministic economy (in which the state alternates between the two values). There are two agents $\{1, 2\}$ with identical utilities $U(c) = E \sum_{t\geq 0} \beta^t u(c_t)$, where u is strictly increasing and concave. At each period t, agent i receives an income $e^i_t := y^H$ if $s_t = i$ and $e^i_t := y^L$ otherwise, with $y^H > y^L$. At any period, the agent with income y^H is referred to as high-type, and the agent with income y^L is the low-type. The penalty for default is the interdiction to trade (3.9).

When agents can trade in one period Arrow securities in zero supply, the stationary equilibria in this framework were studied by Kehoe and Levine (2001) and Alvarez and Jermann (2001). We present these equilibria, support them with infinitely lived assets that dynamically complete the markets (rather than with Arrow securities), and then analyze the effect of bubble injections on trading volume.

There exists a unique stationary equilibrium. For the high (low) type agent, consumption is c^H (c^L), wealth level (beginning of each period) is $-w$ (w), and the unique pricing kernel a is such that $\frac{a_{t+1}}{a_t} = q_c$ if $s_t \neq s_{t+1}$ and $\frac{a_{t+1}}{a_t} = q_{nc}$ if $s_t = s_{t+1}$. Moreover, if the initial levels of wealth do not coincide with the steady state levels, in particular if agents start with no wealth, as we will assume, the transition to the steady state is complete at the first state reversal. During the transition, the agents' consumption is constant, but different from the steady state levels. Steady state consumptions satisfy the market clearing condition $c^L + c^H = y^L + y^H$, and the high-type agent is indifferent between defaulting or not. Letting $\tilde{\beta} = \frac{\beta\pi}{1-\beta(1-\pi)}$, this indifference condition amounts to

$$u(c^H) + \tilde{\beta}u(c^L) = u(y^H) + \tilde{\beta}u(y^L). \tag{4.4}$$

The pricing kernel follows from the Euler conditions of an (unconstrained) high-type,

$$q_{nc} = \beta, \quad q_c = \beta u'(c^L)/u'(c^H). \tag{4.5}$$

Let $\bar{q}_c := \pi q_c$ and $\bar{q}_{nc} := (1 - \pi)q_{nc}$. The beginning of period wealth level for the low-type agent is

$$w = (y^H - c^H)/(1 + \bar{q}_c - \bar{q}_{nc}), \tag{4.6}$$

(and for a high-type is $-w$), while the NTT debt limits for a high-type agent are $\phi^H := -w$ and for a low-type agent are $\phi^L := -\bar{q}_c w/(1 - \bar{q}_{nc})$. As shown in Alvarez and Jermann

(2001), the quantities and prices outlined above are an equilibrium (with imperfect risk sharing) if and only if $y^L < c^L < c^H < y^H$. Bidian and Bejan (2012) show that $y^L < c^L$ and $c^H < y^H$ if and only if $\tilde{\beta}u'(y^L)/u'(y^H) > 1$, and if this assumption holds, then $c^L < c^H$ if and only if

$$(1+\tilde{\beta})u\left(\frac{y^H + y^L}{2}\right) \le u(y^H) + \tilde{\beta}u(y^L). \tag{4.7}$$

Condition (4.7) requires that the first best symmetric allocation in which each agent consumes half of the aggregate endowment does not satisfy the participation constraints of the high-type agents. It can be verified immediately that the price $\bar{q}$ of a riskless asset is less than 1, $\bar{q} := E_t a_{t+1}/a_t = \bar{q}_c + \bar{q}_{nc} < 1$, and therefore interest rates are "high".

We substitute now the Arrow securities with two infinitely lived assets, which dynamically complete the markets. We analyze two types of dividend structure. In the first case, analyzed in the rest of this section, one of the assets pays dividends contingent on a reversal having occurred, and the other way around for the other asset. In the second case, analyzed in Appendix B, asset $j \in \{1, 2\}$ pays dividends at a given period if and only if state j occurred at that period. We assume that agents start with no endowment of securities, hence bubble injections will not affect agents' debt limits (see Theorem 3.2).

There are two infinitely lived assets $\{1, 2\}$ in zero supply with dividends $d_t^1 = \lambda \mathbf{1}_{s_t = s_{t-1}}$, $d_t^2 = \lambda \mathbf{1}_{s_t \neq s_{t-1}}$ for $t > 0$ and equal to zero at $t = 0$, where $\lambda > 0$ and $\mathbf{1}$ is the indicator function (for $A \subset \Omega$ and $\omega \in \Omega$, $\mathbf{1}_A(\omega)$ is 1 if $\omega \in A$ and 0 if $\omega \notin A$). Thus the first asset pays dividends if there is no change in state, while the second asset pays dividends after a reversal of state. Agents start with zero endowments of the assets. We focus on a period $t \geq 1$ *after* the economy has reached steady state, which happens on the first state reversal. Therefore, if the steady state is reached at $T > 0$ (which is an a.s. finite stopping time), then any variable below with a subscript t is to be understood as referring to period $T + t$. The fundamental values of the assets are

$$p_t^1 = \lambda \sum_{s>t} \bar{q}^{s-t-1}(\bar{q}_c \cdot 0 + \bar{q}_{nc} \cdot 1) = \frac{\lambda \bar{q}_{nc}}{1-\bar{q}},\ p_t^2 = \lambda \sum_{s>t} \bar{q}^{s-t-1}(\bar{q}_c \cdot 1 + \bar{q}_{nc} \cdot 0) = \frac{\lambda \bar{q}_c}{1-\bar{q}}.$$

We replicate agents' wealth levels with portfolios of long-lived securities, eliminating the need for Arrow securities. Thus we construct portfolios θ^i for each agent such that, given the asset prices computed before, $(p_t + d_t)\theta_{t-1}^i$ equals $-w$ if $s_t = i$ and w if $s_t \neq i$. Denote by $\bar{\theta}_{j,t-1}$ the holdings of a low-type at $t-1$ of security j. If the state changes from $t-1$ to t, then $(p_t^1 + 0)\bar{\theta}_{1,t-1} + (p_t^2 + \lambda)\bar{\theta}_{2,t-1} = -w$, while if there is no change,

$(p_t^1 + \lambda)\bar{\theta}_{1,t-1} + (p_t^2 + 0)\bar{\theta}_{2,t-1} = w$. Solving this system of equations, we obtain

$$\bar{\theta}_{1,t-1} = w\lambda^{-1}(2\bar{q}_c + 1 - \bar{q}), \quad \bar{\theta}_{2,t-1} = -w\lambda^{-1}(2\bar{q}_{nc} + 1 - \bar{q}).$$

Notice that the asset prices and portfolios are time invariant after the economy reaches the steady state, and we can omit the time subscript.

We consider bubbles ε which do not crash before the steady state is reached (before T), and their value at T is some positive $\bar{\varepsilon}$. After the steady state is reached, they grow at the rate $\varepsilon_c \geq 0$ if there is a state change, respectively $\varepsilon_{nc} \geq 0$ if there is no state change, that is $\varepsilon_t = \varepsilon_{t-1} \cdot \left(\varepsilon_c \mathbf{1}_{s_t \neq s_{t-1}} + \varepsilon_{nc} \mathbf{1}_{s_t = s_{t-1}}\right)$. A value $\varepsilon_c = 0$ implies that the bubble crashes on the first state change, while if $\varepsilon_{nc} = 0$, the bubble crashes if the state does not change (after the steady state is reached). The trading strategy $\Lambda = \Lambda(\varepsilon, p)$ (replicating the bubble growth) follows from

$$(p_t^1 + 0)\Lambda_{t-1}^{11} + (p_t^2 + \lambda)\Lambda_{t-1}^{21} = \varepsilon_{t-1}\varepsilon_c, \; (p_t^1 + \lambda)\Lambda_{t-1}^{11} + (p_t^2 + 0)\Lambda_{t-1}^{21} = \varepsilon_{t-1}\varepsilon_{nc},$$

and therefore

$$\Lambda_{t-1}^{11} = \lambda^{-1}(\varepsilon_{nc} - 1) \cdot \varepsilon_{t-1}; \quad \Lambda_{t-1}^{21} = \lambda^{-1}(\varepsilon_c - 1) \cdot \varepsilon_{t-1}. \tag{4.8}$$

Denote by $SV_t^j(c)$, respectively $SV_t^j(nc)$ the share volumes in asset j if state changes (c), respectively it does not change (nc) from $t-1$ to t, and similarly for dollar volumes, and the share and dollar volumes after the bubble injection. Notice that $SV_t^1(c) = 2\bar{\theta}_1 > 0$, $SV_t^1(nc) = 0$, $SV_t^2(c) = -2\bar{\theta}_2 > 0$, $SV_t^2(nc) = 0$ and

$$\hat{SV}_t^1(c) = \bar{\theta}_1 \left| \frac{1}{1 + \Lambda_{t-1}^{11}} + \frac{1}{1 + \Lambda_t^{11}} \right|, \; \hat{SV}_t^1(nc) = \bar{\theta}_1 \left| \frac{1}{1 + \Lambda_{t-1}^{11}} - \frac{1}{1 + \Lambda_t^{11}} \right|,$$
$$\hat{SV}_t^2(c) = \left| -2\bar{\theta}_2 + \bar{\theta}_1 \frac{\Lambda_{t-1}^{21}}{1 + \Lambda_{t-1}^{11}} + \bar{\theta}_1 \frac{\Lambda_t^{21}}{1 + \Lambda_t^{11}} \right|, \; \hat{SV}_t^2(nc) = \bar{\theta}_1 \left| \frac{\Lambda_{t-1}^{21}}{1 + \Lambda_{t-1}^{11}} - \frac{\Lambda_t^{21}}{1 + \Lambda_t^{11}} \right|.$$

Therefore a bubble injection in the first asset *increases* the share volume of trade in *both* assets at periods when there is *no reversal*, since the volume of trade jumps from zero to a positive value. If the state changes, the effect of a bubble injection depends on the type of bubble introduced.

For a *stochastic* bubble that *crashes on the first reversal* (after the steady state is reached), $\varepsilon_c = 0$, $\varepsilon_{nc} = \bar{q}_{nc}^{-1}$, $\Lambda_{t-1}^{11} = \lambda^{-1}(\varepsilon_{nc} - 1)\varepsilon_{t-1} > 0$, $\Lambda_{t-1}^{21} = -\lambda^{-1}\varepsilon_{t-1}$ (the case of a *deterministic* bubble is analyzed in Appendix B). If there is no reversal from $t-1$ to t, the bubble increases the trading volume in both securities. If t is large, the share volume of trade in both securities is close to zero, but the dollar volume in the first security $\hat{DV}^1_t(nc)$ is bounded away from zero, as it approaches $\lambda\bar{\theta}_1$. If the state changes from $t-1$ to t, the bubble crashes and $\Lambda^{11}_t = \Lambda^{21}_t = 0$. Therefore the share and dollar volume of trade in the first asset decrease in the period when the bubble crashes, as

$$\hat{SV}^1_t(c) = \bar{\theta}_1\left(1 + \frac{1}{1+\Lambda^{11}_{t-1}}\right) < 2\bar{\theta}_1 = SV^1_t(c),$$

and $\hat{DV}^1_t(c) = p^1\hat{SV}^1_t(c) < p^1 SV^1_t(c) = DV^1_t(c)$. The volume of trade in the second security also decreases when the bubble crashes, since it can be checked that

$$\hat{SV}^2_t(c) = \left|-2\bar{\theta}_2 + \bar{\theta}_1\frac{\Lambda^{21}_{t-1}}{1+\Lambda^{11}_{t-1}}\right| < -2\bar{\theta}_2 = SV^2_t(c).$$

In summary, as long as the stochastic bubble is running, the share and dollar volume of trade are higher than normal. The dollar volume of trade in the first security is bounded away from zero. When the bubble collapses, the volume of trade shrinks to levels lower than normal. After the crash, the trading volume reverts back to normal.

We calibrate the example using the parameters already employed by Alvarez and Jermann (2001) in their analysis of the volatility of the pricing kernel in this model. Thus $\beta = 0.65$, $y^H = 0.641$, $y^L = 0.359$, $\pi = 0.25$, and $u(c) = c^{1-\gamma}/(1-\gamma)$, with $\gamma = 2$. Finally, we take $\lambda = 0.03$ as being the average ratio of US net corporate dividends to gross domestic product (GDP) for 1947-2011 (Federal Reserve Economic Data). It follows that $c^H \approx 0.639$, $c^L \approx 0.361$, $\bar{q}_{nc} \approx 0.487$, $\bar{q}_c \approx 0.507$, $p^1 \approx 1.772$, $p^2 \approx 1.843$, $\bar{\theta}_1 \approx 0.124$, $\bar{\theta}_2 \approx -0.12$. The value of the stochastic bubble when the economy enters the steady state is assumed to be $\bar{\varepsilon} = 0.001$. We compare the dollar trade volumes after the bubble with their levels without the bubble. The increase in the dollar volume of trade in the first period if the bubble has not crashed yet is 0.007 for each asset. Therefore a very small initial bubble, equal to 0.1% of the GDP, generates an initial increase in the total trade volume of 14 times its size. Conditional on the bubble not having crashed, the increase in trade volume continues to grow (for 5 periods) and reaches a maximum of 7.81% of GDP, and then tapers off, approaching 0.25% of GDP if the bubble runs for a long time. If the bubble crashes in the first period, the drop in trade volume equals 0.55% of GDP, while if the bubble is sustained for a long time and then crashes, the drop in trade volume is 16.2% of GDP. In relative terms, when compared to the no-bubble case, the total trade volume drops by 1.68% if the bubble crashes in the first period, and by 49.7% if the bubble crashes after a long run. Thus small bubbles can produce disproportionately large increases in the volume of trade, and subsequent large collapses in trade volume when they crash.

In Appendix B, we show, for completeness, that a deterministic (rather than stochastic) bubble also increases the trading volume. Then, as a robustness check, we consider a different dividend structure and show that the qualitative and quantitative effects of bubbles on trading volume are extremely similar.

5 Effects of bubbles on returns

Bubble injections affect asset returns. As in Section 4, we compare the two "equivalent" equilibria of Theorem 3.2, the bubble-free equilibrium $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ and the bubbly equilibrium $\left(\hat{p}, (\hat{w}^i)_{i=1}^I, (c^i, \hat{\theta}^i)_{i=1}^I\right)$. In this way, we disentangle the effects of a bubble on the financial sector from its allocational effects (which we cannot study at this level of generality). The analysis is straightforward, and we explore a few of the asset pricing puzzles that can be addressed via bubbles.

Consider a trading strategy $\theta \in X^{J\times 1}$ with a positive gross return at prices $p, \hat{p}$, that is $R_{t+1} := (p_{t+1} + d_{t+1})\theta_t/(p_t\theta_t) \geq 0$, $\hat{R}_{t+1} := (\hat{p}_{t+1} + d_{t+1})\theta_t/(\hat{p}_t\theta_t) \geq 0$, for all $t \geq 0$. A sufficient condition for the positivity of returns is $\theta \in X_+^{J\times 1}$, which holds in the following discussion, as θ is assumed to represent either a market index, or a buy-and-hold portfolio in an individual stock. Since $\hat{p}\theta = p\theta + \varepsilon\theta$,

$$\hat{R}_{t+1} = \frac{p_t\theta_t}{\hat{p}_t\theta_t}R_{t+1} + \left(1 - \frac{p_t\theta_t}{\hat{p}_t\theta_t}\right)\frac{\varepsilon_{t+1}\theta_t}{\varepsilon_t\theta_t}. \tag{5.1}$$

For example, if we want to bubbly and bubble-free returns on first stock, it is enough to take $\theta_t = (1, 0, \ldots, 0)' \in \mathbb{R}^J$, for all t. In this case the returns on the first asset in the bubbly equilibrium are

$$\hat{R}_{t+1}^1 = \frac{p_t^1}{\hat{p}_t^1}R_{t+1}^1 + \left(1 - \frac{p_t^1}{\hat{p}_t^1}\right)\frac{\varepsilon_{t+1}^1}{\varepsilon_t^1}. \tag{5.2}$$

Therefore the returns in the bubbly equilibrium are an average of the fundamental (bubble-free) returns and the rate of growth of the bubble component. This property makes the analysis of bubble effects immediate.

A bubble increases the risk premium on the index if and only if the bubble growth rate has a higher risk premium (covaries more negatively with the stochastic discount factor). Similarly, the bubble increases the Sharpe ratio of the index if it has a higher Sharpe ratio (is correlated more negatively with the stochastic discount factor). To establish these facts, for any to random variables Z_1, Z_2 that are $\mathcal{F}_{t+1}$-measurable, denote by $Cov_t(Z_1, Z_2)$ and $\rho_t(Z_1, Z_2)$ their conditional covariance and conditional correlation given $\mathcal{F}_t$, and by $\sigma_t(Z_1)$ the conditional standard deviation of Z_1 given $\mathcal{F}_t$.

Proposition 5.1. *Let $a \in A_{++}(p)$ and let $m_{t+1} := a_{t+1}/a_t$ be the associated stochastic discount factor. Let $R^f_{t+1} := 1/E_t m_{t+1}$ be the risk free rate and $R^e_{t+1} := R_{t+1} - R^f_{t+1}$, $\hat{R}^e_{t+1} := \hat{R}_{t+1} - R^f_{t+1}$. The following hold:*

$$E_t\hat{R}^e_{t+1} \geq E_t R^e_{t+1}\ (\geq 0) \Leftrightarrow E_t\frac{\varepsilon_{t+1}}{\varepsilon_t} - R^f_{t+1} \geq E_t R^e_{t+1} - R^f_{t+1}\ (\geq 0)$$

$$\Leftrightarrow Cov_t\left(m_{t+1}, \frac{\varepsilon_{t+1}}{\varepsilon_t}\right) \leq Cov_t(m_{t+1}, R_{t+1})\ (\leq 0). \tag{5.3}$$

$$\frac{E_t\frac{\varepsilon_{t+1}}{\varepsilon_t} - R^f_{t+1}}{\sigma_t\left(\frac{\varepsilon_{t+1}}{\varepsilon_t}\right)} \geq \frac{E_t R^e_{t+1}}{\sigma_t(R_{t+1})} \geq 0 \Leftrightarrow \rho_t\left(m_{t+1}, \frac{\varepsilon_{t+1}}{\varepsilon_t}\right) \leq \rho_t(m_{t+1}, R_{t+1}) \leq 0 \Rightarrow$$

$$\Rightarrow \frac{E_t\hat{R}^e_{t+1}}{\sigma_t(\hat{R}_{t+1})} \geq \frac{E_t R^e_{t+1}}{\sigma_t(R_{t+1})} \geq 0. \tag{5.4}$$

Proof. By (3.3), the return R_{t+1} satisfies $E_t m_{t+1} R_{t+1} = 1$, and therefore its (conditional) risk premium and Sharpe ratio satisfy

$$E_t(R_{t+1} - R^f_{t+1}) = -R^f_{t+1} Cov_t(m_{t+1}, R_{t+1}), \tag{5.5}$$

$$\frac{E_t R_{t+1} - R^f_{t+1}}{\sigma_t(R_{t+1})} = \frac{-R^f_{t+1} Cov_t(m_{t+1}, R_{t+1})}{\sigma_t(R_{t+1})} = -R^f_{t+1}\sigma_t(m_{t+1}) \cdot \rho_t(m_{t+1}, R_{t+1}). \tag{5.6}$$

Similarly, (5.5) and (5.6) also hold when R_{t+1} is replaced by the bubble growth $\frac{\varepsilon_{t+1}\theta_t}{\varepsilon_t\theta_t}$, as $E_t m_{t+1}\frac{\varepsilon_{t+1}\theta_t}{\varepsilon_t\theta_t} = 1$, by (3.5). Now (5.3) follows from (5.5), while (5.4) follows from (5.6) and

$$\sigma_t(\hat{R}_{t+1}) \leq \frac{p_t\theta_t}{\hat{p}_t\theta_t}\sigma_t(R_{t+1}) + \left(1 - \frac{p_t\theta_t}{\hat{p}_t\theta_t}\right)\sigma_t\left(\frac{\varepsilon_{t+1}\theta_t}{\varepsilon_t\theta_t}\right). \tag{5.7}$$

□

Proposition 5.1 shows that bubbles can increase risk premia and Sharpe ratio. However, as implied by Proposition A.2, the return from holding a deflator-preserving bubble is the return on a trading strategy in the existing assets. Together with (5.1), this implies that the bubble inflated return $\hat{R}_{t+1}$ coincides with the return on a portfolio of the initial J assets, and therefore the bubble cannot increase the risk premia and the Sharpe ratio to levels higher than the maximal risk premia and Sharpe ratios than can be achieved by arbitrary trading strategies (leading to positive gross returns).

Conversely, a very large bubble at t can increase the risk premium, respectively Sharpe ratio on the index to levels arbitrarily close to those achieved by portfolios with positive gross returns and maximal risk premium, respectively maximal Sharpe ratio (in the absence of the bubble). Indeed, in this case $p_t\theta_t/(\hat{p}_t\theta_t)$ becomes arbitrarily close to zero, and therefore the return $\hat{R}_{t+1}$ becomes arbitrarily close to $\varepsilon_{t+1}\theta_t/(\varepsilon_t\theta_t)$. Choosing the bubble growth to mimic the return on a portfolio with a large risk premium (Sharpe ratio) leads to a high risk premium (Sharpe ratio) for the bubble inflated returns.[18] An identical discussion applies to kurtosis rather than risk premia or Sharpe ratios, thus a bubble can generate fat tails in an asset, but not more so than the maximal kurtosis of a return on a portfolio of the initial assets.

By (5.1) and (5.7), the conditional risk premium and Sharpe ratio of the bubble-inflated returns varies over time, as the "weight" $p_t/\hat{p}_t$ of the initial return in the bubbly return varies with the size of the bubble. In fact, a countercyclical expected bubble growth (that is, a current boom is associated with a larger bubble and future lower expected bubbles) introduces countercyclical movements in risk premia and Sharpe ratios. The variability of the risk premium and Sharpe ratio over time without an accompanying variability of the volatility of consumption came to be referred to as the *conditional equity premium puzzle* (Cochrane 2000, Chapter 21).

Bubbles create movements in asset prices decoupled from changes in fundamentals. Therefore it is rather obvious that bubbles can introduce additional volatility in asset

[18] Choose $\Lambda \in X^{J\times J}$ such that $\Lambda_t\theta_t$ has a high expected risk premium (Sharpe ratio), in the absence of bubbles. If $\mathbf{I}+\Lambda_t$ is singular, then we perturb slightly Λ_t to avoid the drop in rank problem (see (3.7)), otherwise we leave Λ_t unchanged. Then we set

$$\frac{\varepsilon_{t+1}\theta_t}{\varepsilon_t\theta_t} = \frac{(p_{t+1}+d_{t+1})\Lambda_t\theta_t}{p_t\Lambda_t\theta_t}.$$

prices. For example, for a bubble ε^1 in the first asset,

$$\frac{\sigma^2(\hat{p}_t^1)}{\sigma^2(p_t^1)} = \frac{\sigma^2(p_t^1 + \varepsilon_t^1)}{\sigma^2(p_t)} = 1 + \frac{\sigma(\varepsilon_t^1)}{\sigma(p_t^1)} \left(\frac{\sigma(\varepsilon_t^1)}{\sigma(p_t)} + 2\rho(\varepsilon_t, p_t) \right),$$

and therefore the volatility of prices increases whenever the volatility of the bubble is high enough (it is sufficient to be twice as high as volatility of prices), or if the correlation between the bubble and fundamental prices is high enough (it is sufficient to be positive). An identical argument applies to price-dividend ratios rather than prices. An extended discussion of the *excess volatility puzzle* can be found in Cochrane (2000, Chapter 20).

6 Conclusion

We consider the (large) class of processes that preserve the set of pricing kernels (deflators) when added to asset prices ("deflator-preserving" processes). Any nonnegative such process can be injected as a rational bubble in asset prices, leading to an equilibrium with identical allocations and pricing kernels, but with debt limits tightened proportionally to the size of the bubble. Moreover, with enforcement limitations, if the debt bounds are endogenized as in Alvarez and Jermann (2000) to prevent default but to allow for maximal credit expansion, the modified debt limits in the equilibrium with bubbles still arise endogenously from the existing enforcement limitations.

Any such deflator-preserving bubble acts therefore as a device that relaxes the debt limits of the agents. Bubble collapses/inceptions endogenously reduce/increase the liquidity (credit) available in the economy. Bubbles can cause large increases in trading volume while they run and large collapses upon their crash, compared to trading volumes in the absence of bubbles. The class of bubbles identified here can generate also high and time-varying Sharpe ratios, and excess volatility of asset prices.

A Deflator-preserving processes

We characterize the set $M^J(p)$ of deflator-preserving processes. For each $t \geq 1$, let $\mathcal{S}_t(p)$ be the set of attainable payoffs at t given the price and dividend processes $p, d \in X_+^{1\times J}$:

$$\mathcal{S}_t(p) := \{(p_t + d_t)\lambda \mid \lambda : \Omega \to \mathbb{R}^J \text{ and } \lambda \text{ is } \mathcal{F}_{t-1} - \text{measurable}\}. \tag{A.1}$$

We refer to $\mathcal{S}_t(p)$ as the period t *asset span.*

Lemma A.1. *Let $p, d \in X_+^{1\times J}$ such that $A(p) \neq \emptyset$ and $\varepsilon \in X^{1\times J}$. The following are equivalent:*

(i) There is $\Lambda \in X^{J\times J}$ such that $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$ for all $t \geq 1$ and there is $a \in A(p)$ such that $a \cdot \varepsilon$ is a martingale.

(ii) $\mathcal{S}_t(p+\varepsilon) \subset \mathcal{S}_t(p)$ for all $t \geq 1$ and there is $a \in A(p)$ such that $a \cdot \varepsilon$ is a martingale.

(iii) There is $\Lambda \in X^{J\times J}$ such that $\varepsilon_{t+1} = (p_{t+1} + d_{t+1})\Lambda_t$, $\varepsilon_t = p_t\Lambda_t$, for all $t \geq 0$.

(iv) $A(p) \subset A(p+\varepsilon)$

(v) For each $a \in A(p)$, $a \cdot \varepsilon$ is a martingale.

Proof. (i) $\Leftrightarrow$ (ii) For the implication (i) $\Rightarrow$ (ii), let $(p_t + d_t)\lambda \in \mathcal{S}_t(p+\varepsilon)$, with $\lambda : \Omega \to \mathbb{R}^J$, $\mathcal{F}_{t-1}$-measurable. Then

$$(p_t + \varepsilon + d_t)\lambda = (p_t + d_t)(\mathbf{I} + \Lambda_{t-1})\lambda \in \mathcal{S}_t(p).$$

Conversely, for any $\lambda_{t-1} : \Omega \to \mathbb{R}^J$ which is $\mathcal{F}_{t-1}$-measurable, there exists $\lambda'_{t-1} : \Omega \to \mathbb{R}^J$, $\mathcal{F}_{t-1}$-measurable, such that $(p_t + d_t + \varepsilon_t)\lambda_{t-1} = (p_t + d_t)\lambda'_{t-1}$. It follows that $\varepsilon_t\lambda_{t-1} = (p_t + d_t)(\lambda'_{t-1} - \lambda_{t-1})$, and since λ_{t-1} was arbitrary, we conclude that each of the J components of ε_t belongs to $\mathcal{S}_t(p)$. Thus $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$ for some $\mathcal{F}_{t-1}$-measurable $\Lambda_{t-1} : \Omega \to \mathbb{R}^{J\times J}$.

(i) $\Rightarrow$ (iii) The conclusion is immediate, since for all $t \geq 0$,

$$\varepsilon_t = E_t \frac{a_{t+1}}{a_t}\varepsilon_{t+1} = E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1})\Lambda_t = p_t\Lambda_t.$$

(iii) $\Rightarrow$ (iv) Let $a \in A(p)$. The conclusion follows, since

$$E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1} + \varepsilon_{t+1}) = E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1})(\mathbf{I} + \Lambda_t) = p_t(\mathbf{I} + \Lambda_t) = p_t + \varepsilon_t.$$

(iv) $\Rightarrow$ (v) Let $a \in A(p)$. Thus $a \in A(p+\varepsilon)$. The conclusion follows, since

$$p_t + E_t \frac{a_{t+1}}{a_t}\varepsilon_{t+1} \overset{a\in A(p)}{=} E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1} + \varepsilon_{t+1}) \overset{a\in A(p+\varepsilon)}{=} p_t + \varepsilon_t, \forall t \geq 0.$$

(v) $\Rightarrow$ (i) Assume that $m \in X$ is such that $a \cdot m$ is a martingale, for any $a \in A(p)$. Fix a date t and state $\omega \in \Omega$, and denote by $\mathcal{F}_t(\omega)$ the atom of the partition $\mathcal{F}_t$ containing ω

(that is, the date t "node" containing ω, in the tree language). Assume that $\mathcal{F}_{t+1}$ has S atoms belonging to $\mathcal{F}_t(\omega)$ (i.e. there are S branches leaving the chosen node). Then the returns r_{t+1} conditional on the event $\mathcal{F}_t(\omega)$ can be viewed as an $S \times J$ matrix R. Similarly m_{t+1}/m_t conditional on $\mathcal{F}_t(\omega)$ is represented by a vector $M \in \mathbb{R}^S$. If $\mu \in \mathbb{R}^S$ is interpreted as conditional state price process a_{t+1}/a_t times conditional probabilities, it follows that for any $\mu \in \mathbb{R}^S$ such that $\mathbf{1}' = \mu' R$, it must be the case that $1 = \mu' M$. Therefore there cannot exist $\mu \in \mathbb{R}^S$ such that

$$\begin{cases} \mu'(R^1 - M) < 0 \\ \mu'(R^1 - R^j) = 0, \quad j \in \{2, \ldots, J\}. \end{cases}$$

By Motzkin's alternative theorem (Motzkin 1951), there exist $\alpha_2, \ldots, \alpha_J \in \mathbb{R}$ such that $R^1 - M = \sum_{j=2}^J \alpha_j (R^1 - R^J)$. Therefore M can be written as a linear combination of the columns of R and there exists $\lambda \in X^{J \times 1}$ such that $m_t = (p_t + d_t)\lambda_{t-1}$, for all $t \geq 1$.

Each component ε^j of $\varepsilon = (\varepsilon^1, \ldots, \varepsilon^J)$ is a martingale when deflated by any $a \in A(p)$. As proven above, for each j there exists $\lambda^j \in X^{J \times 1}$ such that $\varepsilon^j_t = (p_t + d_t)\lambda^j_{t-1}$ for all $t \geq 1$. The conclusion follows by setting $\Lambda = (\lambda^1, \ldots, \lambda^J)$. □

We say that *there are no redundant securities* at $t-1$, given prices p, if there is no $\lambda : \Omega \to \mathbb{R}^J$ such that λ is $\mathcal{F}_{t-1}$−measurable, $\lambda \neq 0$ and $(p_t + d_t)\lambda = 0$. We give several equivalent characterizations of the set $M^J(p)$ of deflator-preserving processes.

Proposition A.2. *The following are equivalent:*

(i) $\varepsilon \in M^J(p)$.

(ii) $a \cdot \varepsilon$ *is a martingale, for any* $a \in A(p) \cup A(p + \varepsilon)$

(iii) $\mathcal{S}_t(p) = \mathcal{S}_t(p + \varepsilon)$*, for all* $t \in \mathbb{N}$*, and* $a \cdot \varepsilon$ *is a martingale, for some* $a \in A(p)$.

(iv) There exists $\Lambda, \Gamma \in X^{J \times J}$ *such that for all* $t \geq 0$,

$$\varepsilon_t = p_t \Lambda_t = (p_t + \varepsilon_t)\Gamma_t, \quad \varepsilon_{t+1} = (p_{t+1} + d_{t+1})\Lambda_t = (p_{t+1} + \varepsilon_{t+1} + d_{t+1})\Gamma_t.$$

Moreover, $\bar{M}^J(p)$ *(see (3.7)) is a subset of* $M^J(p)$ *and if there are no redundant securities at any period* t*,* $\bar{M}^J(p) = M^J(p)$.

Proof. (i) $\Leftrightarrow$ (ii) Follows from the equivalence (iv) $\Leftrightarrow$ (v) in Lemma A.1. $A(p) \subset A(p+\varepsilon)$ is equivalent to $a \cdot \varepsilon$ is a martingale for any $a \in A(p)$. Similarly, $A(p+\varepsilon) \subset A(p+\varepsilon+(-\varepsilon))$ is equivalent to $a \cdot (-\varepsilon)$ (hence $a \cdot \varepsilon$) being a martingale for all $a \in A(p+\varepsilon)$.

(i) $\Rightarrow$ (iii) $A(p) \subset A(p+\varepsilon)$ implies $\mathcal{S}_t(p+\varepsilon) \subset \mathcal{S}_t(p)$ and $a \cdot \varepsilon$ is a martingale, for some $a \in A(p)$ (Lemma (A.1), (i) $\Leftrightarrow$ (ii)). Similarly, $A(p+\varepsilon) \subset A(p+\varepsilon+(-\varepsilon))$ gives $\mathcal{S}_t(p) \subset \mathcal{S}_t(p+\varepsilon)$.

(iii) $\Rightarrow$ (iv) Using $a \in A(p)$ and $a \cdot \varepsilon$ martingale, it follows that $a \in A(p+\varepsilon)$. The conclusion follows from Lemma A.1, (i) $\Leftrightarrow$ (iii), using $\mathcal{S}_t(p+\varepsilon) \subset \mathcal{S}_t(p)$ and $\mathcal{S}_t(p+\varepsilon+(-\varepsilon)) \subset \mathcal{S}_t(p+\varepsilon)$.

(iv) $\Rightarrow$ (i) By Lemma (A.1), (iii) $\Leftrightarrow$ (iv), it follows that $A(p) \in A(p+\varepsilon)$ and $A(p+\varepsilon) \subset A(p+\varepsilon+(-\varepsilon))$.

For the last part, notice that if $\varepsilon \in \bar{M}^J(p)$ and $\Lambda \in \Lambda(p,\varepsilon)$,

$$\varepsilon_{t+1} = (p_{t+1} + d_{t+1})\Lambda_t = (p_{t+1} + \varepsilon_{t+1} + d_{t+1})(\mathbf{I} + \Lambda_t)^{-1}\Lambda_t.$$

Similarly, $\varepsilon_t = p_t\Lambda_t = (p_t + \varepsilon_t)(\mathbf{I} + \Lambda_t)^{-1}\Lambda_t$. Choosing $\Gamma := (\mathbf{I} + \Lambda_t)^{-1}\Lambda$, the equivalence (iv) $\Rightarrow$ (i) (just established above) shows that $\varepsilon \in M^J(p)$.

Finally, assume that there are no redundant securities and $\varepsilon \in M^J(p)$. Then the set $\Lambda(\varepsilon, p)$ is a singleton. From part (iv), there exist Λ, Γ such that $\varepsilon_{t+1} = (p_{t+1} + d_{t+1})\Lambda_t = (p_{t+1}+d_{t+1})(\mathbf{I}+\Lambda_t)\Gamma_t$ and $\varepsilon_t = p_t\Lambda_t = p_t(\mathbf{I}+\Lambda_t)\Gamma_t$. Hence $(\mathbf{I}+\Lambda)\Gamma \in \Lambda(\varepsilon, p)$, and therefore $(\mathbf{I} + \Lambda)\Gamma = \Lambda$. It follows that

$$\mathbf{I} = \mathbf{I} + \Lambda - (\mathbf{I} + \Lambda)\Gamma = (\mathbf{I} + \Lambda)(\mathbf{I} + \Gamma),$$

and we conclude that $\mathbf{I} + \Lambda$ is non-singular and $\varepsilon \in \bar{M}^J(p)$. □

$M^J(p)$ is therefore also the set of processes ε that are a martingale when discounted by any deflator in $A(p)$ and $A(p+\varepsilon)$. Equivalently, $M^J(p)$ is also the set of all discounted martingales (under some $a \in A(p)$) that preserve the asset span if added to asset prices. Finally, $M^J(p)$ is the set of processes ε with rates of growth replicable by returns on portfolios under the initial prices p and the adjusted prices $p + \varepsilon$ (if the process is added to asset prices). Thus there are some portfolios $\Lambda = (\Lambda^1, \ldots, \Lambda^J) \in X^{J \times J}$ and $\Gamma =$

$(\Gamma^1, \ldots, \Gamma^J) \in X^{J\times J}$ such that

$$\frac{\varepsilon^j_{t+1}}{\varepsilon^j_t} = \frac{(p_{t+1}+d_{t+1})\Lambda^j_t}{p_t\Lambda^j_t} = \frac{(\hat{p}_{t+1}+d_{t+1})\Gamma^j_t}{\hat{p}_t\Gamma^j_t}, \forall t \geq 0, \forall j \in \{1,\ldots,J\},$$

where $\hat{p} = p + \varepsilon$. Finally, the last part of the proposition shows that if there are no redundant securities, than the (unique) portfolio Λ replicating the rate of growth of ε (at prices p) can be used to construct a portfolio Γ replicating the rate of growth of ε at prices $p+\varepsilon$, and therefore making the structure of the set $M^J(p)$ substantially more transparent.

B Extensions of the example in Section 4

For completeness, we show that a deterministic (rather than stochastic) bubble also increases the trading volume in the example of Section 4. Then we consider a different dividend structure and show that the qualitative and quantitative effects of bubbles on trading volume are extremely similar.

For a *deterministic* bubble, $\varepsilon_c = \varepsilon_{nc} = \bar{q}^{-1}$, $\Lambda^{11}_{t-1} = \Lambda^{21}_{t-1} = \lambda^{-1}(1-\bar{q})\bar{q}^{-t}\bar{\varepsilon}$. If there is no reversal from $t-1$ to t, the bubble increases the volume of trade in both securities, as seen before. However, asymptotically these increases vanish except for the dollar volume of trade in first asset. Indeed, for large t, since $\Lambda^{11}_t \nearrow \infty$, it follows that $\hat{SV}^1_t(nc), \hat{SV}^2_t(nc), \hat{DV}^2_t(nc) \approx 0$, while $\hat{DV}^1_t(nc) \approx \lambda\bar{\theta}_1$. When there is a reversal from $t-1$ to t, the share volume of trade in the first security decreases, while it increases for the second security. The dollar volume of trade increases however even for the first security,

$$\hat{DV}^1_t(c) = \left(p^1_t + \varepsilon_t\right)\hat{SV}^1_t(c) = \left(p^1_t(1+\Lambda^{11}_t) + p^2_t\Lambda^{21}_t\right)\hat{SV}^1_t(c) > p^1_t \cdot 2\bar{\theta}_1 = DV^1_t.$$

For large t, $\hat{SV}^1_t(c) \approx 0$, $\hat{SV}^2_t(c) \approx 2\bar{\theta}_1 - 2\bar{\theta}_2$, $\hat{DV}^2_t(c) \approx 2p^2(\bar{\theta}_1 - \bar{\theta}_2)$, while $\hat{DV}^1_t(c) \approx \lambda\bar{\theta}_1(1+\bar{q})/(1-\bar{q})$. Thus a deterministic bubble in the first asset always increases the dollar volumes of trade in both assets. The increase in the dollar volume of trade in the first asset is persistent.

We check next the robustness of the conclusions obtained in the example in Section 4 by assuming that dividends depend on the current state (rather than whether a reversal occurred). The dividends of the two securities are $d^i_t = \lambda\mathbf{1}_{s_t=i}$ for $t > 0$, and zero at $t = 0$. Thus asset $j \in \{1, 2\}$ pays dividends λ at t if state j is realized at t, and zero otherwise. It is immediate to see that asset prices depend only on the realization of the current state, thus $p_t^j = p^j(s_t)$. The fundamental valuation equation gives

$$p^1(1) = \bar{q}_c\left(p^1(2)+0\right) + \bar{q}_{nc}\left(p^1(1)+\lambda\right),\ p^1(2) = \bar{q}_c\left(p^1(1)+\lambda\right) + \bar{q}_{nc}\left(p^1(2)+0\right),$$

hence

$$p^1(1) = \frac{1}{2}\cdot\frac{\lambda\bar{q}}{1-\bar{q}} + \frac{1}{2}\cdot\frac{\lambda(\bar{q}_{nc}-\bar{q}_c)}{1-(\bar{q}_{nc}-\bar{q}_c)},\ p^1(2) = \frac{1}{2}\cdot\frac{\lambda\bar{q}}{1-\bar{q}} - \frac{1}{2}\cdot\frac{\lambda(\bar{q}_{nc}-\bar{q}_c)}{1-(\bar{q}_{nc}-\bar{q}_c)}.$$

By symmetry, $p^2(1) = p^1(2)$, $p^2(2) = p^1(1)$. Let $\theta_{t-1}^i(k)$ denote the portfolio of agent i at $t-1$ if the state realized at $t-1$ is k. It follows that

$$(p^1(1)+\lambda)\theta_{t-1}^{1,1}(1) + p^2(1)\theta_{t-1}^{1,2}(1) = -w, \quad p^1(2)\theta_{t-1}^{1,1}(1) + (p^2(2)+\lambda)\theta_{t-1}^{1,2}(1) = w,$$

and therefore $\theta_{t-1}^{1,1}(1) = -\theta_{t-1}^{1,2}(1) = -w/(p^1(1)-p^1(2)+\lambda) < 0$. A similar reasoning shows that $\theta_{t-1}^{1,1}(2) = -\theta_{t-1}^{1,2}(2) = \theta_{t-1}^{1,1}(1)$. Since the steady state portfolios are time invariant and do not depend on the state process, we can *drop* the time subscripts and the state arguments. The agents hold balanced amounts of the two securities, equal in absolute value, but of opposite sign. The share and dollar volume of trade in both securities are zero (after the steady state is reached), $SV_t^j = DV_t^j = 0$, $j \in \{1,2\}$. Therefore an (arbitrary) bubble injection *increases* the share and dollar volume of trade in *all* securities.

Consider first a *deterministic* bubble (ε_t) (in the first asset). The process $\Lambda \in X^{2\times 1}$ satisfying $(p_t+d_t)\Lambda_{t-1} = \varepsilon_t$ is $\Lambda_{t-1}^{11} = \Lambda_{t-1}^{21} = \bar{\varepsilon}\bar{q}^{-t}(p^1(1)+p^2(1)+\lambda)^{-1}$. The volume of trade t periods *after* the economy reaches the steady state is

$$\begin{aligned}
\hat{SV}_t^1 &= \left|\frac{1}{1+\Lambda_t^{11}}\theta^{1,1} - \frac{1}{1+\Lambda_{t-1}^{11}}\theta^{1,1}\right| = |\theta_t^{1,1}|\frac{\Lambda_t^{11}(1-\bar{q})}{(1+\Lambda_t^{11})(1+\bar{q}\Lambda_t^{11})} \to_{t\to\infty} 0,\\
\hat{SV}_t^2 &= \left|\theta^{1,2} - \frac{\Lambda_t^{21}}{1+\Lambda_t^{11}}\theta^{1,1} - \theta^{1,2} + \frac{\Lambda_{t-1}^{21}}{1+\Lambda_{t-1}^{11}}\theta^{1,1}\right| = \hat{SV}_t^1 \to_{t\to\infty} 0,\\
\hat{DV}_t^1 &= (p^1(s_t)+\bar{\varepsilon}\bar{q}^{-t})\hat{SV}_t^1 \to_{t\to\infty} |\theta^{1,1}|(1-\bar{q})(p^1(1)+p^1(2)+\lambda).
\end{aligned}$$

It follows that the market-wide increase in the share volume of trade induced by a deterministic bubble (in the first asset) vanishes asymptotically, while the increase in the dollar volume of trade in the first asset is persistent.

The effects of a *stochastic* bubble can be analyzed in a similar fashion. Consider (as in Section 4) a bubble in the first asset that *crashes on the first reversal* (after the steady state is reached). For concreteness, assume that the economy starts in state 2 and therefore the steady state is reached when the state switches to 1 for the first time. Thus $\varepsilon_c = 0$, $\varepsilon_{nc} = \bar{q}_{nc}^{-1}$. The bubble spanning portfolios satisfy

$$(p^1(1) + \lambda)\Lambda_{t-1}^{11} + p^2(1)\Lambda_{t-1}^{21} = \varepsilon_{nc}\varepsilon_{t-1}, \quad p^1(2)\Lambda_{t-1}^{11} + (p^2(2) + \lambda)\Lambda_{t-1}^{21} = 0,$$

and therefore

$$\Lambda_{t-1}^{11} = \frac{\varepsilon_{t-1}\bar{q}_{nc}^{-1}(p^1(1) + \lambda)}{(p^1(1) + \lambda)^2 - (p^1(2))^2}, \quad \Lambda_{t-1}^{21} = -\frac{\varepsilon_{t-1}\bar{q}_{nc}^{-1}p^1(2)}{(p^1(1) + \lambda)^2 - (p^1(2))^2}.$$

It can be checked immediately that Λ_{t-1}^{11} is positive and grows at the rate $\bar{q}_{nc}^{-1} > 1$ as long as the bubble does not crash. As was the case for the deterministic bubble, the share volume of trade in both securities approaches zero if the bubble runs for a long time, but the dollar volume of trade in the first asset is bounded away from zero, and approaches $|\theta^{1,1}|(1 - \bar{q}_{nc})((p^1(1) + \lambda)^2 - (p^1(2))^2)/(p^1(1) + \lambda)$.

With the numerical calibration of the previous section, $p^1(1) \approx 2.711$, $p^1(2) \approx 2.712$, $\theta^{1,1} \approx -0.083$. A stochastic bubble (crashing on the first reversal) of size 0.1% of the GDP at the period when the economy reaches the steady state increases the total trade volume by 1.5% of GDP in the first period. The trade volume increase continues to grow initially, reaching a maximum of 7.97% of GDP after 5 periods, and then starts to drop, but nevertheless the increase is persistent and approaches 0.25% of GDP if the bubble runs for a long time.

C Trading volume in an example with incomplete markets

The uncertainty is described by a time homogeneous Markov process $(s_t)_{t\in\mathbb{N}}$ with $s_t \in \{1, 2, 3\}$, having a transition probability matrix π with strictly positive entries. There are two agents $\{1, 2\}$ with utilities $U^i(c) = E\sum_{t\geq 0} \beta^t u(c_t)$, where u is strictly increasing and strictly concave. There are two assets in unit supply. In each period, the first asset pays $y > 0$ if the current state is 1, $y/2$ in state 2, and 0 in state 3. The second asset pays 0 in state 1, $y/2$ in state 2, and y in state 3. We assume we have a security markets economy, in that agents' only income is generated by dividends resulting from their asset holdings. Agent $i \in \{1, 2\}$ has an initial endowment of security i equal to 1, and a zero endowment of the other security. The agents face zero debt limits.[19]

We construct a Pareto optimal equilibrium in which expected (gross) returns are equal to β^{-1} and agents have constant consumption. Thus the securities are fairly priced in that their price equals the expected value of dividends discounted at the risk free rate β^{-1}. There will be no trade after the initial period, when the portfolios are adjusted once and for all. This is not surprising in light of the result in Judd, Kubler, and Schmedders (2003) described before, and emphasizes that the absence of trade is a consequence of Pareto optimality (rather than of complete markets).

Asset prices are the present value of future dividends,

$$p_t^j = E_t \sum_{s>t} \beta^{s-t} d_s^j, \quad \forall j \in \{1,2\}, \forall t \geq 0. \tag{C.1}$$

Beginning of period wealth levels are obtained from the intertemporal budgets,

$$(p_t + d_t)\theta_{t-1}^i = E_t \sum_{s \geq t}^{\infty} \beta^{s-t} c^i = \frac{c^i}{1-\beta}, \quad \forall i \in \{1,2\}, \forall t \geq 0. \tag{C.2}$$

By writing (C.2) at $t = 0$ we obtain the (constant) consumption levels,

$$c^i = (1-\beta)(p_0^i + d_0^i) = (1-\beta) \sum_{t=0}^{\infty} \beta^t \cdot E d_t^i, \quad \forall i \in \{1,2\}. \tag{C.3}$$

Notice that $p_t^1 + p_t^2 = \beta y/(1-\beta)$ and $p_t^1 + d_t^1 + p_t^2 + d_t^2 = y/(1-\beta)$. Therefore, generically in π, (C.2) admits only the solution

$$\theta_t^{i,1} = \theta_t^{i,2} = \frac{c^i}{y}, \quad \forall i \in \{1,2\}, \forall t \geq 0. \tag{C.4}$$

To check that the allocations, portfolios and prices described in (C.1)-(C.4) form an equilibrium, it is enough to prove that the consumptions and portfolios are optimal for each agent, as the market clearing conditions are clearly satisfied. But this is true, since the given consumptions and portfolios satisfy the necessary and sufficient Kuhn-Tucker and

[19] These, of course, are NTT when the penalty for default is the interdiction to borrow (3.10). Since agents' wealth originates solely from financial wealth, these debt limits are also given by (3.11), where debt is restricted by the present value of future endowments.

transversality conditions for agents' utility maximization problems:

$$p_t = E_t \frac{\beta u'(c^i)}{u'(c^i)}(p_{t+1} + d_{t+1}), \forall t \geq 0, \text{ and } \lim_{t\to\infty} E\beta^t u'(c^i) p_t \theta_t^i = 0.$$

Given the zero volume of trade following the initial period, *arbitrary* bubble injections *increase* the share and dollar volume of trade.[20] To analyze further the volume of trade effects, we focus for concreteness on a *deterministic* bubble with an initial value $\bar{\varepsilon}$, injected in the first asset. Thus $\varepsilon_t = \beta^{-t}\bar{\varepsilon}$. The bubble spanning portfolios are generically (in π) unique and given by

$$\Lambda_{t-1}^{11} = \Lambda_{t-1}^{21} = \bar{\varepsilon}\beta^{-t}y^{-1}(1-\beta). \tag{C.5}$$

By (4.3) and (C.4) the share volume of trade in both assets increases, but this increase vanishes asymptotically, as $-\frac{\Lambda_t^{21}}{1+\Lambda_t^{11}} = \frac{1}{1+\Lambda_t^{11}} - 1$ and

$$S\hat{V}_t^1 = |\hat{\theta}_t^{i,1} - \hat{\theta}_{t-1}^{i,1}| = \left| \frac{\theta_t^{i,1} - \theta_{-1}^{i,1}}{1+\Lambda_t^{11}} - \frac{\theta_{t-1}^{i,1} - \theta_{-1}^{i,1}}{1+\Lambda_{t-1}^{11}} \right| = \frac{\theta_t^{2,1}(1-\beta)\Lambda_t^{11}}{(1+\Lambda_t^{11})(1+\beta\Lambda_t^{11})} \to_{t\to\infty} 0,$$

$$S\hat{V}_t^2 = |\hat{\theta}_t^{i,2} - \hat{\theta}_{t-1}^{i,2}| = \left| \left(\frac{1}{1+\Lambda_t^{11}} - 1\right)\theta_t^{i,1} - \left(\frac{1}{1+\Lambda_{t-1}^{11}} - 1\right)\theta_{t-1}^{i,1} \right| = S\hat{V}_t^1 \to_{t\to\infty} 0.$$

However the increase in the dollar volume of trade in the first asset is persistent, as

$$D\hat{V}_t^1 = (p_t^1 + \varepsilon_t) S\hat{V}_t^1 \to \lim \bar{\varepsilon}\beta^{-t} S\hat{V}_t^1 = c^2.$$

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[20]However, due to constant expected returns, the expected rate of growth of bubble coincides with the expected return on any asset, hence the equity premium is unaffected by the presence of bubbles, and remains zero.

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