A bargaining-Walras equivalence for finite economies and further characterizations

Javier Hervés-Estévez
RGEA. Universidad de Vigo.
e-mail: javiherves@uvigo.es

Emma Moreno-García
Universidad de Salamanca.
e-mail: emmam@usal.es

Abstract. We give a notion of bargaining set for economies with a finite number of agents. We state an equivalence theorem showing that this bargaining set coincides with the set of Walrasian allocations. Our bargaining-Walras equivalence provides a discrete approach to the characterization of competitive equilibria obtained by Mas-Colell (1989) for continuum economies. Moreover, we also show that justified objections equate with Walrasian objections.

Some further results highlight whether it is possible to restrict the formation of coalitions and still get the bargaining set. Finally, recasting some known characterizations of Walrasian allocations we state additional interpretations of the bargaining set.

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1 Introduction

The core of an economy is defined as the set of allocations that can not be blocked by any coalition. The veto mechanism that defines the core is based on a single move and it does not take into account any other consequence this move may have. Thus, one may ask whether this objection or veto is credible or, on the contrary, not consistent enough so other agents in the economy may react to it and propose a re-blocking or counter-objection.

The first outcome of this two-step conception of the veto mechanism was the work by Aumann and Maschler (1964) who introduced the concept of bargaining set, containing the core of a cooperative game. This original concept of bargaining set was adapted later to atomless economies by Mas-Colell (1989). The main idea is to try to inject a sense of stability to the veto mechanism, and hence permitting the implementation of some allocations that otherwise would be formally blocked. Thus, only objections that can not be counter-objected (the credible, the stable ones) are allowed, and blocking an allocation becomes more difficult. In other words, the objections which have no counter-objections are precisely those that are considered as justified.

In the case of pure exchange economies with a finite number of traders it is well known that the set of Walrasian allocations is a strict subset of the core which is also strictly contained in the bargaining set. Under conditions of generality similar to those required in Aumann's (1964) core equivalence theorem, Mas-Colell (1989) shows that the bargaining set and the competitive allocations coincide for continuum economies. These types of equivalence results provide foundations for the Walrasian market equilibrium and, at the same time, arise the fundamental question of whether the equivalence results have analogies in economies with large, but a finite number of agents. A classical contribution in this direction is that by Debreu and Scarf (1963), who state a first formalization of Edgeworth's (1881) conjecture, showing that the core and the set of Walrasian allocations become arbitrarily close whenever a finite economy is replicated sufficiently many times. However, in contrast with the Debreu-Scarf core convergence theorem, the work by Anderson, Trockel and Zhou (1997) proves that the bargaining set does not shrink to the set of Walrasian allocations in a sequence of replicated economies as the core does.[1]

[1] The replica sequence in the example stated by Anderson, Trockel and Zhou (1997) satisfies the hypotheses of the Debreu-Scarf theorem (1963); preferences are smooth and the economy

Therefore, in contrast to the core, the Mas-Colell bargaining set does not lead to a convergence result in large finite economies. Roughly speaking, this is basically due to the fact that the notion of a justifiable objection is very stringent. Thus, given the difficulties to find such credible objections, the bargaining set may become very large and even include most of the feasible allocations. The example stated by Anderson, Trockel and Zhou (1997) highlights this point. Precisely, they define a sequence of replica economies in which there is a unique Walrasian equilibrium but the bargaining set eventually occupies the full measure of the set of all individual rational and Pareto optimal allocations having the equal treatment property.

Nevertheless, as Anderson, Trockel and Zhou (1997) point out, the argument supporting their non-convergence example depends crucially on the use of a replica structure to enlarge the economy. Indeed, the lack of convergence in their example is completely driven by an integer numbers problem. Consequently, they leave open the possibility that other ways of enlarging the set of agents and, in turn, strengthening the blocking power of coalitions in the economy, might lead to other results.

Addressing finite economies, Aubin (1979) has proposed a veto mechanism where agents can participate in coalitions with a part of their endowments, and has showed that the core resulting from this blocking system equals the set of Walrasian allocations. It is important to realize that this veto *à la Aubin* represents actually a way of enlarging the set of coalitions. Furthermore, the Aubin core-Walras equivalence leads us to consider the Aubin veto to define objections and counter-objections. In this way, we define a concept of bargaining set for finite economies where it becomes less demanding to have justified objections. Then, we prove the main result of our present work: the set of Walrasian allocations coincides with this bargaining set in economies with a finite number of traders. That is, we obtain an equivalence result which provides a finite approach to the characterization obtained by Mas-Colell (1989) of competitive allocations.

This equivalence result opens up the opportunity to provide additional results of different nature. For instance, as an immediate consequence, it allows us to claim that the bargaining set we have defined for finite economies is consistent in the sense of Dutta *et al.* (1989) as it happens with the Mas-Colell bargaining set for atomless economies.

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is regular.

Furthermore, we provide a discrete approach to the characterization of justified objections made by Mas-Colell (1989) by means of Walrasian objections. Although Mas-Colell uses this characterization to prove his main equivalence theorem, it has value in itself. In our case, having that any Walrasian objection is justified and vice-versa for finite economies, allows us to refine our bargaining-Walras equivalence and its proof in terms of Walrasian objections.

Our bargaining-Walras equivalence (and also Mas-Colell's) implicitly requires the formation of all coalitions in the objecting and counter-objecting process. In other words, the bargaining set concept demands to check the whole set of possible coalitions in order to test whether any group of agents can improve upon an allocation by using their own resources, either in the objecting or counter-objecting process. It is usually argued that the costs arising from forming a coalition are not at all negligible: incompatibilities among different agents may appear and a big amount of information and communication might be needed to really get together a coalition. This idea leads us to study the possibility of restricting the formation of coalitions by assuming that not all the parameters, which specify the degree of participation of agents when they become members of a coalition, are admissible. Then, we analyze the consequences that this condition has with regard to the bargaining set solution. We show that both for objections and counter-objections, the participation rates of the agents can be restricted to those arbitrarily small without changing the bargaining set. However, we point out with examples that this does not hold if we consider parameters close enough to the complete participation.

Finally, we try to make the best use of our main equivalence result by recasting in terms of the bargaining set some characterizations of the Walrasian allocations present throughout the literature. First, we focus on a result by Hervés-Beloso, Moreno-García and Yannelis (2005) that characterizes Walrasian allocations as those that are not blocked by the coalition formed by all the agents in a collection of perturbed economies. Then, we revisit the approach followed by Hervés-Beloso and Moreno-García (2009) who show that Walrasian equilibria can be identified by using a non-cooperative two-player game. Both equivalence theorems constitute now additional characterizations of the bargaining set for finite economies.

The rest of the work is structured as follows. In Section 2 we collect notations and preliminaries. Next, in Section 3, a Walras-bargaining set equivalence result for finite economies is provided. Section 4 contains a characterization of justified objections via Walrasian objections for finite economies. Section 5 elaborates on the possibility of restricting the coalitions that are allowed to form to get the bargaining set. In Section 6, specific equivalence theorems for Walrasian equilibrium are presented as further characterizations of the bargaining set. In order to facilitate the reading of the paper, the proofs of the results are contained in a final Appendix.

2 Preliminaries

Let $\mathcal{E}$ be an exchange economy with a finite number n of agents who trade a finite number ℓ of commodities. Each consumer i has a preference relation $\succsim_i$ on the set of consumption bundles $\mathbb{R}^\ell_+$, with the properties of continuity, convexity and monotonicity. This means that preferences are represented by utility functions $U_i \;\; i \in \{1, \ldots, n\}$. Let $\omega_i \in \mathbb{R}^\ell_{++}$ denote the endowments of consumer i. So the economy is $\mathcal{E} = (\mathbb{R}^\ell_+; U_i; \omega_i)_{i \in \{1,\ldots,n\}}$.

An allocation x is a consumption bundle $x_i \in \mathbb{R}^\ell_+$ for each agent $i = 1, \ldots, n$. The allocation x is feasible in the economy $\mathcal{E}$ if $\sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i$. A price system is an element of the $(\ell - 1)$-dimensional simplex of $\mathbb{R}^\ell_+$. A Walrasian equilibrium for the economy $\mathcal{E}$ is a pair (p, x), where p is a price system and x is a feasible allocation such that, for every agent i, the bundle x_i maximizes the utility function U_i in the budget set $B_i(p) = \{y \in \mathbb{R}^\ell_+ \text{ such that } p \cdot y \leq p \cdot \omega_i\}$. We denote by $W(\mathcal{E})$ the set of Walrasian allocations for the economy $\mathcal{E}$.

A coalition is a non-empty set of consumers. An allocation y is said to be attainable or feasible for the coalition S if $\sum_{i \in S} y_i \leq \sum_{i \in S} \omega_i$. Let $x \in \mathbb{R}^{\ell n}_+$ be an allocation in the economy $\mathcal{E}$. The coalition S blocks x if there exists an allocation y which is attainable for S, such that $U_i(y_i) \geq U_i(x_i)$ for every $i \in S$ and $U_j(y_j) > U_j(x_j)$ for some member j in the coalition S. A feasible allocation x is efficient if it is not blocked by the big coalition $N = \{1, \ldots, n\}$ formed by all the agents. The core of the economy $\mathcal{E}$, denoted by $C(\mathcal{E})$, is the set of feasible allocations which are not blocked by any coalition of agents.

It is well known that under the hypotheses above the economy $\mathcal{E}$ has Walrasian equilibrium, and any Walrasian allocation belongs to the core (in particular, it is efficient). It is also known that the blocking power of coalitions in finite economies is not able to eliminate every non-Walrasian allocation. Then, in order to characterize the Walrasian equilibria in terms of the cooperative solution of the core we have to enlarge the set of coalitions or, alternatively, increase somehow their veto power. This line of arguments has been carried out following different ways. For instance, Aubin (1979) extends the notion of ordinary veto to a weighted veto mechanism for finite economies. The blocking scheme introduced by Aubin gives more power to coalitions allowing the agents to participate with a portion of their endowments when forming a coalition. We refer to this veto system as *Aubin veto* or *veto in the sense of Aubin*.

An allocation x is blocked in the sense of Aubin by the coalition S via the allocation y if there exist $\alpha_i \in (0,1]$, for each $i \in S$, such that (i) $\sum_{i\in S} \alpha_i y_i \leq \sum_{i\in S} \alpha_i \omega_i$, and (ii) $U_i(y_i) \geq U_i(x_i)$, for every $i \in S$ and $U_j(y_j) > U_j(x_j)$ for some $j \in S$. The Aubin core of the economy $\mathcal{E}$, denoted by $C_A(\mathcal{E})$, is the set of all feasible allocations which cannot be blocked in the sense of Aubin. Under the standard assumptions stated above, Aubin (1979) showed that any Walrasian allocation is in the Aubin core, and, reciprocally, any non-Walrasian allocation is blocked in the sense of Aubin.

Both the core and the Aubin core depend on the notion of a coalition objecting to a proposed allocation but neglect to take into account the repercussions triggered by the improvement moves. Thus, these cooperative solutions in economic theory exclude allocations to which there exist objections but does not assess the "credibility" of such objections. This kind of thought has led Aumann and Maschler (1964) to establish the definition of bargaining set within a cooperative game theory framework. Recently, this classical bargaining set has been extended by Yang, Liu and Liu (2011) to Aubin bargaining sets in a set of games that they refer to as convex cooperative fuzzy games. Shortly after, Liu and Liu (2012) give a modification of the previous extension and obtain both existence and equivalence results with other cooperative solutions. However, as they have remarked, finding a most reasonable way to such extensions is not a trivial matter.

After Aumann and Machler (1964) provided the original bargaining set notion for cooperative games, several versions have been defined and studied for exchange economies. Specifically, Mas-Colell (1989) was the first to define the bargaining set for economies with a continuum of agents.[2] The idea of the def-

[2]Mas-Colell (1989) not only has adapted the original concept of bargaining set to atomless economies but also has proved, under conditions of generality similar to the Aumann's (1964) core equivalence theorem, that the bargaining set and the set of competitive allocations

inition is that this set contains all the feasible allocations of the economy that are not blocked or objected by any coalition and, additionally, it also contains the ones that are blocked or objected in a non-credible way. In other words, an allocation belongs to the bargaining set whenever either it belongs to the core or any objection to it is not sustainable because it results in a counter-objection.

To precise the notion of Mas-Colell bargaining set for the finite economy $\mathcal{E}$, let x be a feasible allocation that is blocked by a coalition S via the allocation y. Then, the objection (S, y) to x has a counter-objection if there exists a coalition T and a attainable allocation z for T such that $U_i(z_i) > U_i(y_i)$ for every $i \in T \cap S$ and $U_i(z_i) > U_i(x_i)$ for every $i \in T \setminus S$, where $T \setminus S$ is the set of agents which are in T but not in S.

An objection which can not be counter-objected is said to be justified. Thus, the Mas-Colell bargaining set of an economy contains all the feasible allocations that, if they are objected (or blocked) they could also be counter-objected by using the usual veto mechanism. Let $B_{MC}(\mathcal{E})$ denote the Mas-Colell bargaining set for the economy $\mathcal{E}$ with n consumers.

In the next section, we provide foundations of Walrasian equilibrium via a bargaining set. For this, we will consider the already mentioned veto system proposed by Aubin instead of the usual blocking mechanism. Thus, we extend and adapt the notions of the bargaining sets recently provided by Yang, Liu and Liu (2011) and Liu and Liu (2012) for (transferable utility) cooperative games to finite exchange economies. In addition, we will use the fact that, regarding Walrasian equilibria, a finite economy $\mathcal{E}$ with n consumers is equivalent to a continuum economy $\mathcal{E}_c$ with n-types of agents as we specify in which follows.

Consider a continuum economy where the set of agents is represented by the unit real interval $[0, 1]$ endowed wit the Lebesgue measure μ (as in Aumann, 1964). There are only a finite number of types of consumers. Thus, $I = [0, 1] = \bigcup_{i=1}^{m} I_i$, with $\mu(I_i) = n_i/n$ (i.e., $\mu(I_i)$ is a rational number).[3] Every $t \in I_i$ has the same endowments ω_i and utility function U_i, that is, all the consumers in I_i are of the same type i. Note that we can write $I_i = \bigcup_{j=1}^{n_i} I_{ij}$ with $\mu(I_{ij}) = 1/n$ for every i, j. Consider now a finite economy with n agents and n_i consumers of each type i. Note that a feasible allocation $x = (x_1, \ldots, x_n)$, with $x_i = (x_{ij}, j = 1, \ldots, n_i)$,

coincide.

[3]Without loss of generality one can take $I_i = [a_i, a_{i+1})$, for every $i \in \{1, \ldots, m-1\}$; with $a_1 = 0$, $a_{i+1} - a_i = n_i/n$ and $I_m = [a_m, 1]$. Equivalently, we can also take $I = [0, n]$ and $I_i = [n_i, n_i + n_{i+1})$, for every $i \in \{1, \ldots, m-1\}$; with $n_1 = 0$ and $I_m = [n_m, n]$.

in the finite economy defines a feasible allocation f_x in the continuum economy which is given by $f_x(t) = x_{ij}$ for every $t \in I_{ij}$. Reciprocally, a feasible allocation f in the continuum economy defines a feasible allocation x^f in the finite economy which is given by $x^f_{ij} = \frac{1}{\mu(I_{ij})} \int_{I_{ij}} f(t) d\mu(t)$. Moreover, x (resp. f) is an equal-treatment allocation if and only if so is f_x (resp. x^f).

Under continuity and convexity of preferences, García-Cutrín and Hervés-Beloso (1993) showed that if (x, p) is a Walrasian equilibrium in the n-agent economy, then (f_x, p) is a competitive allocation in the n-types continuum economy. Conversely, if (f, p) is a competitive equilibrium in the continuum economy then (x^f, p) is a Walrasian equilibrium in the finite economy.

Consider now the economy $\mathcal{E}$ that we have define at the beginning of this section. Let $\mathcal{E}_c$ be the associated continuum economy, where the set of agents is $I = [0, 1] = \bigcup_{i=1}^{n} I_i$, where $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right)$ if $i \neq n$; $I_n = \left[\frac{n-1}{n}, 1\right]$; and all the agents in the subinterval I_i are of the same type i. In this particular case, $x = (x_1, \ldots, x_n)$ is a Walrasian allocation in the finite economy $\mathcal{E}$ if and only if the step function f_x (defined by $f_x(t) = x_i$ for every $t \in I_i$) is a competitive allocation in the continuum economy $\mathcal{E}_c$. In short, the initial finite economy $\mathcal{E}$ and the associated continuum economy $\mathcal{E}_c$ are equivalent regarding market equilibrium.

3 A bargaining-Walras equivalence for finite economies

In economies with a continuum of agents that trade a finite number of commodities, the competitive equilibrium is not only characterized by the core (Aumann, 1964), but also by the bargaining set (Mas-Colell, 1989). As we have already remarked, these characterizations do not hold in finite economies with a finite number of agents, where the set of Walrasian allocations is strictly contained in the core. Moreover, the Mas-Colell bargaining set, which is well defined for finite economies, can be larger than the core (see example in section 6 in Mas-Colell, 1989). Nevertheless, the classical convergence result by Debreu and Scarf (1963) shows that the core shrinks to the set of Walrasian allocations when the economy is enlarged via replicas.

Since models with a continuum of agents are thought of as idealizations of large economies, it could seem reasonable to expect that the Mas-Colell bargaining set would become approximately competitive in sequences of finite exchange economies as the number of agents increases. However, Anderson, Trockel and Zhou (1997) show that the bargaining set does not shrink to the set of Walrasian allocations by replicating the economy. They state a replica sequence of economies where the Mas-Colell bargaining sets do not converge no matter how nice the preferences may be.[4] In the example, that satisfies the hypotheses of the Debreu-Scarf theorem, the measure of the set of individually rational Pareto optimal equal-treatment (IRPOET) allocations which are not in the bargaining set tends to zero as the economy is replicated. In particular, the set formed by all the IRPOET allocations which belong to the bargaining set converges in the Hausdorff distance to the set of all IRPOET allocations.

Thus, the above cited work by Anderson, Trockel and Zhou (1997) gives insights into the discrepancy between the behavior of the Mas-Colell bargaining set in the continuum and its behavior in sequences of large finite economies. In addition, the authors explicitly remark that their non-convergence example is driven entirely by an integer problem. Moreover, they also observe that their argument does make significant use of the replica structure, which leaves open other possibilities to have better behaved bargaining sets.

Actually, as we will show in this section, the adjustment of the definition of objection and counter-objection by using the veto mechanism *à la Aubin*, allows us to characterize the set of Walrasian allocations via the bargaining set.

An (Aubin) objection to x in the economy $\mathcal{E}$ is a pair (S, y), where S is a coalition that blocks x via y in the sense of Aubin. Note that the coalition S can be also defined by the parameters which specify the participation of its members.

A (Aubin) counter-objection to the objection (S, y) is a pair (T, z), where T is a coalition and z is an allocation defined on T, for which there exist $\lambda_i \in (0, 1]$ for each $i \in T$, such that:

(i) $\sum_{i\in T} \lambda_i z_i \leq \sum_{i\in T} \lambda_i \omega_i$,

(ii) $U_i(z_i) > U_i(y_i)$ for every $i \in T \cap S$ and

(iii) $U_i(z_i) > U_i(x_i)$ for every $i \in T \setminus S$

[4] Precisely, they provide a non-convergence result for Zhou (1994) bargaining set, which requires additional restrictions on counter-objections. These restrictions make justified objections easier to form and then make the bargaining set smaller than the Mas-Colell one.

Definition 3.1 *A feasible allocation belongs to the (Aubin) bargaining set of the finite economy if it has no justified objection* à la Aubin. *In other words, if every objection to the allocation has also a counter-objection using the Aubin veto system.*

We denote by $B(\mathcal{E})$ the bargaining set of the economy $\mathcal{E}$ as we have defined above. We remark that, by definition, any allocation in the Aubin core of the economy belongs to the bargaining set. Thus, $W(\mathcal{E}) = C_A(\mathcal{E}) \subseteq B(\mathcal{E})$.

From now on, and unless stated otherwise, every time we are in a finite economy framework and write block, objection, counter-objection, or any other concept related with a veto system, we refer to those notions in the sense of Aubin. On the other hand, whenever we address continuum economies the blocking mechanism we consider is the standard one.

To clarify, let us highlight the main differences between the Mas-Colell bargaining set $B_{MC}(\mathcal{E})$ and the one we have stated *à la Aubin* $B(\mathcal{E})$. In our definition agents can join a coalition, in both objecting and counter-objecting process, with a part of their initial endowments. In other words, regarding the bargaining system, agents can cooperate with different participation levels and the attainable bundles depend on these degrees of involvement. Furthermore, whenever an agent i has been assigned the commodity bundle y_i within a coalition involved in an objection, if she also joins a coalition for a counter-objection, then necessarily needs to be assigned a bundle that improves her upon y_i, independently of the rate of participation of agent i in the coalition.[5] We stress that the Aubin veto mechanism is the usual one in the replicated economies as long as the participation rates are fractions and equal-treatment allocations are considered in the replicas. In this case, if the participation in coalitions can be only rational numbers, the requirements to have a counter-objection following the notion we have defined are stronger than in Mas-Colell's solution.

Therefore, the notion of bargaining set we provide strengthens the blocking power of the coalitions and, at the same time, the recasting of the parameters of participation in coalitions as the blocking system in the sequence of replicated economies, with equal treatment allocations, turns the counter-objections into a more demanding process. Essentially, we can conceive that our bargaining set

[5]This remark provides a different way to overcome the weakness (pointed out by Liu and Liu, 2012) of the related fuzzy bargaining set introduced by Yang, Liu and Liu (2011) for (transferable utility) cooperative games.

notion requires a leadership condition concerning the types of agents that are involved in the objection system. Thus, the bargaining set we consider in this paper constitutes indeed an adequate way of "enlarging" the economy and hence permits us to provide the next main theorem, which states a bargaining-Walras equivalence for economies with a finite number of consumers.

Theorem 3.1 *The bargaining set of the finite economy* $\mathcal{E}$ *coincides with the set of Walrasian allocations.*

We stress the sketch of the proof, which is included in the Appendix. As we have already pointed out, it is immediate that any Walrasian allocation belongs to the bargaining set. To show the converse, we consider a feasible allocation x in the finite economy $\mathcal{E}$. We prove that if the step function f_x has an objection with no counter-objection in the associated continuum economy, then there is an (Aubin) objection to x in $\mathcal{E}$ which is no (Aubin) counter-objected either. That is, if f_x is not a competitive allocation in $\mathcal{E}_c$, equivalently, f_x does not belong to $B_{MC}(\mathcal{E}_c)$, then x is not in $B(\mathcal{E})$. Actually, we show more than we say. From the proof of our bargaining-Walras equivalence result we can also state the following:

If (S, g) is an objection to f_x in $\mathcal{E}_c$, then $(\bar{S}, \bar{g})$ is an objection to x in $\mathcal{E}$, where $\bar{S} = \{i \in \{1, \dots, n\} \mid \mu(S_i) = \mu(S \bigcap I_i) > 0\}$ and $\bar{g}_i = \frac{1}{\mu(S_i)} \int_{S_i} g(t) d\mu(t)$, for every $i \in \bar{S}$. Moreover, if (S, g) is a justified objection to f_x in $\mathcal{E}_c$, then $(\bar{S}, \bar{g})$ is a justified objection to x in $\mathcal{E}$.

To finish this Section, we remark that Dutta *et al.* (1989) introduced the concept of consistency regarding the bargaining set. Their idea is to go one step further in the bargaining set concept and try to assess not only the credibility of the objections, but also of the counter-objections involved in the process. They establish a notion of consistent bargaining set meaning that each objection in a "chain" of objections is tested (credible) in precisely the same way as its predecessor. However, the authors recognize that in a context of an exchange economy with a continuum of agents, the equivalence result by Mas-Colell (1989) implies that his bargaining set is consistent. Since we provide an equivalence result for an exchange economy with a finite number of agents, we also overcome this issue, and therefore the notion of bargaining set that we define is also consistent.

4 Justified objections as Walrasian objections

In the case of continuum economies, Mas-Colell (1989) shows that justified objections are characterized by the so-called Walrasian objections. The concept of Walrasian objection requires the introduction of a price system p, and is based on a self selection property: agents in a coalition that participate in a Walrasian objection against an allocation are those ones that would rather trade at the price vector p than get the consumption bundle allocated to them by such an allocation. Thus, Mas-Colell (1989) characterizes justified objections in atomless economies as those ones that can be price supported, showing that every Walrasian objection is justified and vice-versa.[6]

Next we try to proceed likewise and provide a discrete approach to the aforementioned characterization of justified objections. For it, we consider the weighted veto mechanism that has allowed us to obtain the bargaining-Walras equivalence for finite economies in the previous section.

Definition 4.1 *Let x be an allocation in the finite economy $\mathcal{E}$. An (Aubin) objection (S, y) to x is said to be Walrasian if there exists a price system p such that*

(i) $p \cdot v \geq p \cdot \omega_i$ *if* $U_i(v) \geq U_i(y_i)$, $i \in S$ *and*

(ii) $p \cdot v \geq p \cdot \omega_i$ *if* $U_i(v) \geq U_i(x_i)$, $i \notin S$.

We remark that, under the assumptions of monotonicity and strict positivity of the endowments, we know that $p \gg 0$, and therefore conditions *(i)* and *(ii)* above can be written, respectively, as follows:

$U_i(v) > U_i(y_i)$ implies $p \cdot v > p \cdot \omega_i$, for $i \in S$ and

$U_i(v) > U_i(x_i)$ implies $p \cdot v > p \cdot \omega_i$ for $i \notin S$

Observe that the notion of Walrasian objection in the finite economy $\mathcal{E}$ does not depend explicitly on the rates of participation of the members in the coalition that objects an allocation in the sense of Aubin. Precisely, in order to check

[6] It is known that any non-Walrasian allocation could be improved upon (even strictly) by a coalition using an Walrasian allocation for the coalition (see Townsend, 1983, and Mas-Colell, 1985). However, note that the existence of a Walrasian objection is a stronger property.

whether the objection (S, y) to an allocation in $\mathcal{E}$ is a Walrasian objection, it does not matter the degrees of participation of the individuals joining the coalition S that make the allocation y attainable *à la Aubin*; which becomes important is the set of consumers that are involved in the objection.

Proposition 4.1 *Let x be a feasible allocation in the finite economy $\mathcal{E}$. Then, any objection to the allocation x is justified if and only if it is a Walrasian objection.*

The equivalence obtained by Mas-Colell (1989) is proved by combining the two following steps. First, it is shown that any Walrasian objection is justified and then it is obtained that any non-competitive allocation in the continuum economy has a Walrasian objection against it. Consequently, if an allocation is not competitive, it has a justified objection, and in turn it does not belong to the Mas-Colell bargaining set. This is precisely the argument that Mas-Colell follows to show that, for continuum economies, his bargaining set is contained in the set of competitive allocations, which is enough to conclude that both sets coincide.

The fact that any Walrasian objection is a justified objection in finite economies allows us to refine our bargaining-Walras equivalence and its proof in terms of Walrasian objections. To see this, let x be a feasible allocation in $\mathcal{E}$. Note that we can ensure now that if x is not a Walrasian allocation, then it has a Walrasian objection. Moreover, if (S, g) is a Walrasian objection to f_x in the associated n-types continuum economy $\mathcal{E}_c$, then $(\bar{S}, \bar{g})$ is a Walrasian objection to x in the finite $\mathcal{E}$, where $\bar{S} = \{i \in \{1, \ldots, n\} \mid \mu(S_i) = \mu(S \bigcap I_i) > 0\}$ and $\bar{g}_i = \frac{1}{\mu(S_i)} \int_{S_i} g(t) d\mu(t)$, for every $i \in \bar{S}$.

We stress that, since we actually have that justified and Walrasian objections coincide, one can conclude that such a characterization points out that the concept of Walrasian objection in the finite framework is also more than a technical tool to refine the bargaining-Walras equivalence.

5 Restricting coalition formation

As we have remarked the veto *à la Aubin* represents actually a way of enlarging the set of coalitions. This procedure of strengthening the blocking power of coalitions in both objecting and counter-objecting mechanism has allowed us to obtain a bargaining-Walras equivalence which provides a finite approach to the characterization of competitive allocations obtained by Mas-Colell (1989).

Both Mas Colell's result and our bargaining-Walras equivalence implicitly require the formation of all coalitions in the objecting and counter-objecting process. That is, checking whether a given allocation belongs to the bargaining set seems to require to contemplate the whole set of possible coalitions in order to test whether any group of agents, by using their own resources, can improve upon an allocation either in the objecting or counter-objecting process. This may be a great task, even when the economy is small, provided that agents can participate in a coalition with a part of their endowments. Indeed, the Aubin veto system in a finite economy is equivalent to the blocking scheme in the associated continuum economy, with a finite number of types, conducted by equal-treatment allocations.

We also remark that the formation of coalitions may imply some theoretical difficulties. In fact, it is usually argued that the costs, which arise from forming a coalition, are not at all negligible. Incompatibilities among different agents may appear and a big amount of information and communication might be needed to really form a coalition. Thus, sometimes, it will not suffice to merely say that several agents constitute a coalition since it may result in high formation costs, commitments and constraints, which make difficult to assume that the veto mechanism underlying cooperative solutions, as the core or the bargaining set, works freely and spontaneously.

In this Section, the difficulty to argue that coalition formation is costless leads us to consider a restricted veto mechanism in the procedure leading to the bargaining set. Thus, we assume that not all the parameters, which specify the degree of participation of agents when become members of a coalition, are admissible. Then, we study the consequences that this assumption has with regard to the bargaining set solution.

In order to state a meaningful analysis of the restricted bargaining set, we consider that a coalition S is defined by the rates of participation of its members, which is given by a vector $\lambda_S = (\lambda_i, i \in S) \in (0,1]^{|S|}$, where $|S|$ denotes the cardinality of S.

Consider that for each coalition S the participation rates are restricted to a subset $\Lambda_S \subset [0,1]^{|S|}$. Let us denote by $B_\Lambda(\mathcal{E})$ (resp. $B^\Lambda(\mathcal{E})$) the bargaining set where a coalition S can object (resp. counter-object) only with participation rates in Λ_S. When the set on coalitions is restricted in the objection (resp. counter-objection) process it becomes harder to block an allocation (resp. to counter-obtect an objection) and then we have $B^{\Lambda}(\mathcal{E}) \subseteq B(\mathcal{E}) \subseteq B_{\Lambda}(\mathcal{E})$. In addition, if $\Lambda, \widehat{\Lambda}$ are such that $\Lambda_S \subset \widehat{\Lambda}_S$ for every coalition S, then $B^{\Lambda}(\mathcal{E}) \subseteq B^{\widehat{\Lambda}_S}(\mathcal{E}))$ but $B_{\widehat{\Lambda}}(\mathcal{E}) \subseteq B_{\Lambda}(\mathcal{E})$. Therefore, restricting the set of coalitions that are able to object enlarge the bargaining set whereas restricting the coalitions formation in the counter-objection mechanism diminishes the bargaining set instead. This is so because when not all the coalitions can take part in the bargaining mechanism, on the one hand blocking is harder but on the other hand it is easier that an admissible objection becomes credible or justified.

In the case of continuum economies, following Schmeidler (1972), we can interpret the measure of a coalition as the amount of (or cost of) information and communication needed in order to form such a coalition. Then, may be meaningful to consider those coalitions whose size converges to zero; that is, the coalitions that do not involve high costs to be formed. We convey this argument to economies with a finite number of agents where the veto system in the sense of Aubin is considered. For this, given $\delta \in (0,1]$, let δ-$B(\mathcal{E})$ denote the bargaining set of the economy $\mathcal{E}$ where the participation rate of any agent in any coalition, both in the objecting and counter-objecting procedure, is restricted to be less or equal than δ.

Next result, which is straightforward, is related with the remark on the core of atomless economies stated by Schmeilder (1972), who showed that in order to obtain the core of a continuum economy it is enough to consider the blocking power of arbitrarily small coalitions.

Lemma 5.1 *All the δ-bargaining sets are equal and coincide with the bargaining set in the finite economy $\mathcal{E}$. That is, δ-$B(\mathcal{E}) = B(\mathcal{E})$, for every $\delta \in (0,1]$.*

The above result is in contrast with the work by Schjødt and Sloth (1994) who show that, in continuum economies, when one restricts the coalitions which can enter into objections and counter-objections mechanism to those whose size is arbitrarily small, then the Mas-Colell bargaining set becomes strictly larger than the original one. In other words, in atomless economies and contrary to the core, the formation of only arbitrarily small coalitions in the bargaining process does not allow to characterize the competitive allocations. This is due to the fact that limiting the size of coalitions in continuum economies avoids to obtain justified objections. This is not the case in economies with a finite number of agents when one restricts the participation rates of members forming a coalition to those arbitrarily small.

Symmetrically to Schmeidler's (1972) core characterization for atomless economies, Vind (1972) showed that in order to block any non-competitive allocation it is enough to consider the veto power of arbitrarily large coalitions. This result allows to show that in order to obtain the Aubin core it suffices the formation of only one coalition, namely, the big coalition, which is formed by all the agents in the economy; moreover, for every consumer the endowment participation rate can be chosen to be arbitrarily close to one, i.e., the parameters defining the degree of joining in the big coalition can be restricted to those close to the total participation (see Hervés-Beloso and Moreno-García, 2001 and Hervés-Beloso, Moreno-García and Yannelis, 2005). As it might be not surprising, the next example shows that this restriction on coalition formation can not be adapted to the bargaining set solution we address.

Example 1. Let $\mathcal{E}$ be an economy with two consumers, 1 and 2, who trade two commodities, x and y. Both agents have the same preference relation represented by the utility function $U(x, y) = xy$, and both are initially endowed with one unit of each commodity. Let us consider the feasible allocation A which assigns the bundle $A_1 = (2, 2)$ to the individual 1 and the bundle $A_2 = (0, 0)$ to individual 2. The allocation A does not belong to the bargaining set (it does not belong to the core and it is not a Walrasian allocation). In fact, A is blocked in the sense of Aubin by $S = \{2\}$ with any participation rate $\lambda \in (0, 1]$. Moreover every objection $(\{2\}, (1, 1))$, with any $\lambda \in (0, 1]$, to the allocation A has no counter-objection a la Aubin and, therefore, is justified.

Note that there exists B such that the coalition $\{1, 2\}$ objects A in the sense of Aubin via $B = (B_1, B_2)$, with strictly positive weights. That is, there exits $(\lambda_1, \lambda_2) \in (0, 1]^2$ such that $\lambda_1 B_1 + \lambda_2 B_2 \leq (\lambda_1 + \lambda_2)(1, 1)$. In addition $U(B_1) \geq 4$ and $U(B_2) \geq 0$, with at least one strict inequality. This implies that $U(B_2) < U(\omega_2) = 1$

Therefore, any objection where the participation parameters are restricted to be strictly positive for every consumer is counter-objected by individual 2.

We conclude that in contrast to the Aubin core, we can not restrict the coalition formation to the big coalition with parameters close enough to the total participation. The idea of the example relies on the procedure mechanism that defines the bargaining set.

As we have pointed out, in the previous example, consumer 2 blocks the allocation A via the bundle $(1,1)$. Obviously, in this case, any degree of participation of the agent 2 results in a justified objection. This fact makes out again one of the main conceptual differences between Mas-Colell bargaining set and the bargaining set using the veto mechanism in the sense of Aubin. Precisely, considering the notion of Mas-Colell bargaining set either in replicated economies or n-types continuum economies, if a coalition with a justified objection includes only part of some type of agents then it is not possible for these agents to strictly improve at the objection.[7] This is not the case with our notion of justified objections. Basically, this contrast is due to the somehow leadership condition that a type obtains whenever any agent of such a type takes part in an objection, independently of the degree of participation. Therefore, in particular, if we have a justified objection *à la Aubin* (S, y) to the allocation x in a finite economy, with rates of participation $\lambda_i, i \in S$, then the pair $(\tilde{S}, \tilde{y})$ given by any coalition $\tilde{S}$ such that $\mu(\tilde{S} \bigcap I_i) = \lambda_i$ and $\tilde{y}(t) = y_i$ for every $t \in \tilde{S} \bigcap I_i$, is an objection to the step allocation f_x in the associated continuum economy, although it is not necessarily a justified objection.

We have shown that, in contrast to the Aubin core, the participation of the members that join a coalition in order to object an allocation cannot be restricted to those arbitrarily close to 1 in the process that leads to the bargaining set. Next we state a similar example showing that we cannot state such a restriction in the counter-objecting mechanism either.

Example 2. Let $\mathcal{E}$ be an economy with three consumers, 1, 2 and 3, who trade two commodities, x and y. All the agents have the same preference relation represented by the utility function $U(x,y) = xy$, and are initially endowed with one unit of each commodity. Let us consider the feasible allocation A which assigns the bundle $A_1 = (3,3)$ to the individual 1 and the bundle $A_2 = A_3 = (0,0)$ to individuals 2 and 3. The allocation A is blocked in the sense of Aubin by $S = \{2\}$ with any participation rate $\lambda \in (0,1]$. However, $(\{3\},(1,1))$ is a counter-objection to the objection $(\{2\},(1,1))$. In spite of this, there is no counter-objection to $(\{2\},(1,1))$ if all the participation rates are required to be, for

[7] For more details, see Remark 5 in Mas-Colell (1989). See also the related Lemma 3.5 in Anderson *et al.* (1994)

instance, larger than $1/2$.[8] To see this, assume that $\{1, 2, 3\}$ counter-objects, with weights $\lambda_i, i = 1, 2, 3$. Given the preference relations, we can conclude that $3\lambda_1+\lambda_2 < \lambda_1+\lambda_2+\lambda_3$. We get a contradiction with the fact that $\lambda_1, \lambda_3 \in (1/2, 1]$.

6 Some Remarks

Given our bargaining-Walras equivalence, any characterization of Walrasian equilibrium for finite economies turns immediately into an additional characterization of the bargaining set. In this Section, we pick up two different ways of identifying Walrasian allocations and recast them in terms of bargaining sets as corollaries.

First, let us consider a feasible allocation $x = (x_1, \ldots, x_n)$ in the economy $\mathcal{E}$. Following Hervés-Beloso, Moreno-García and Yannelis (2005), we define a family of economies denoted by $\mathcal{E}(a, x)$, $a = (a_1, \ldots, a_n) \in [0, 1]^n$, which coincide with $\mathcal{E}$ except for the endowments that are $\omega_i(a, x) = a_i x_i+(1-a_i)\omega_i$, for each $a_i \in [0, 1]$. An allocation z (feasible or not) is dominated in an economy if it is blocked by the grand coalition $N = \{1, \ldots, n\}$.

In the aforementioned work it was proved that, under the assumptions we have considered, an allocation x is Walrasian in the economy $\mathcal{E}$ if and only if it is not dominated in any perturbed economy $\mathcal{E}(a, x)$, which allows us to write the next corollary as an immediate consequence of the bargaining-Walras equivalence we have obtained.

Corollary 6.1 *An allocation x belongs to the bargaining set of $\mathcal{E}$ if and only if it is not dominated in any economy $\mathcal{E}(a, x)$.*

An alternative way of stating the above result is: *The allocation x has a justified objection (equivalently, a Walrasian objection) against it in the finite economy $\mathcal{E}$ if and only if x is blocked by the grand coalition in some perturbed economy $\mathcal{E}(a, x)$.*

The essence of second characterization of Walrasian equilibrium that we recast for bargaining sets differs substantially from the previous ones. It follows a non-cooperative game theoretical approach and provides insights into the mechanism through which the bargaining process is conducted.

[8] The same remains true if the parameters are required to be larger than any number in $(1/2, 1)$.

Given a finite economy $\mathcal{E} = (\mathbb{R}^{\ell}_{+}; u_i; \omega_i)_{i \in \{1,\ldots,n\}}$, let us define an associated game $\mathcal{G}$ as follows. There are two players. The strategies sets for the players are denoted by S_1 and S_2 and are given by:

$S_1 = \{ x = (x_1, ..., x_n) \in \mathbb{R}^{\ell n}_{+}$ such that $x_i \neq 0$ and $\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} \omega_i \}$.

$S_2 = \{(a, y) \in [\alpha, 1]^n \times \mathbb{R}^{\ell n}_{+}$ such that $\sum_{i=1}^{n} a_i y_i \leq \sum_{i=1}^{n} a_i \omega_i \}$,

where α is a real number such that $0 < \alpha < 1$.

Given a strategy profile $s = (x, a, y) \in S = S_1 \times S_2$, the payoff functions Π_1 and Π_2, for player 1 and 2, respectively, are defined as follows:

$$\Pi_1(x, a, y) = \min_i \{U_i(x_i) - U_i(y_i)\}$$
$$\Pi_2(x, a, y) = \min_i \{ a_i (U_i(y_i) - U_i(x_i))\}$$

Note that if $\Pi_2(x, a, y) > 0$, then the allocation x is blocked via y by the big coalition being a_i the participation rate of each consumer i. Actually, player 2 gets a positive payoff if and only the big coalition objects in the sense of Aubin the allocation proposed by player 1.

As an immediate consequence of our bargaining-Walras equivalence and Theorem 4.1 in Hervés-Beloso and Moreno-García (2009) we obtain the following corollary.

Corollary 6.2 *x belongs to the bargaining set of the economy $\mathcal{E}$, if and only if $(x, \mathbf{b}, x)$ with $\mathbf{b}_i = b$, for every $i = 1, \ldots, n$, (for instance $(x, \mathbf{1}, x)$) is a Nash equilibrium for the game $\mathcal{G}$.*

As our bargaining-Walras equivalence, the above result relies on the veto mechanism proposed by Aubin. To finish, we remark that the spirit of the bargaining set solution we have considered for finite economies seems to indicate that additional and finer characterizations for the bargaining set could be obtained through non-cooperative solutions of different games, in which a player represents the objection system whereas other one is in charge of the counter-objecting mechanism. For it, more work is needed and is part of our further research.

Appendix

Proof of Theorem 3.1. Since the Aubin core coincides with the set of Walrasian allocations for the economy $\mathcal{E}$ (see Aubin, 1979), we have that any Walrasian allocation has no objection in the sense of Aubin and therefore belongs to the bargaining set of $\mathcal{E}$.

Let us show that $B(\mathcal{E}) \subseteq W(\mathcal{E})$. For it, consider an allocation $x \in B(\mathcal{E})$ and the step function[9] f_x which is a feasible allocation in the associated n-types continuum economy $\mathcal{E}_c$. It suffices to show that f_x belongs to the Mas-Colell bargaining set of $\mathcal{E}_c$.[10]

Indeed, let us assume that f_x is objected by (S, g) (otherwise the proof would be finished) meaning that: $\int_S g(t)d\mu(t) \leq \int_S \omega(t)d\mu(t)$, $U_t(g(t)) \geq U_t(f_x(t))$ for every $t \in S$ and $\mu\left(\{t \in S | U_t(g(t)) > U_t(f_x(t))\}\right) > 0$.

Let $S_i = S \cap I_i$ and $\bar{S} = \{i \in \{1, \ldots, n\} | \mu(S_i) > 0\}$. Since S blocks f_x via g we have that there exists a type $k \in \{1, \ldots, n\}$ and a set $A \subset S_k = S \cap I_k$, with $\mu(A) > 0$, such that $U_k(g(t)) > U_k(f_x(t))$, for every $t \in A$.

Let $\bar{g}$ be the allocation given by $\bar{g}_i = \frac{1}{\mu(S_i)} \int_{S_i} g(t)d\mu(t)$, for every $i \in \bar{S}$. Then, by convexity[11] of the preferences, we have $U_k(\bar{g}_k) > U_k(x_k) = U_k(f_x(t))$ for every $t \in S_k$.[12] Note that we also have $U_i(\bar{g}_i) \geq U_i(x_i) = U_i(f_x(t))$ for every $t \in S_i = S \cap I_i$ and $i \in \bar{S}$.

Note that we have constructed an objection $(\bar{S}, \bar{g})$ *à la Aubin* to the allocation x in the economy $\mathcal{E}$, since we have that

(i) $\sum_{i \in \bar{S}} \mu(S_i)\bar{g}_i \leq \sum_{i \in \bar{S}} \mu(S_i)\omega_i$,

(ii) $U_i(\bar{g}_i) \geq U_i(x_i)$ for every $i \in \bar{S}$ and

(iii) there exists $k \in \bar{S}$ such that $U_k(\bar{g}_k) > U_k(x_k)$.

[9]For every $t \in [0, 1]$, $f_x(t) = x_i$ if $t \in I_i$

[10]This is so because the Mas-Colell bargaining set of $\mathcal{E}_c$ equals the set of competitive allocations (Mas-Colell, 1989), which is also equivalent to the core (Aumann, 1964), and f_x is competitive in $\mathcal{E}_c$ if and only if x is Walrasian in $\mathcal{E}$ (García-Cutrín and Hervés-Beloso, 1993).

[11]The convexity of preferences we require is the following: If a consumption bundle a strictly preferred to b so is the convex combination $\lambda a + (1 - \lambda)b$ for any $\lambda \in (0, 1)$. This convexity property is weaker than strict convexity and it holds, for instance, when the utility functions are concave. In such a case, we can apply Jensen's inequality.

[12]See Lemma in García-Cutrín and Hervés-Beloso (1993) for further details.

Since the allocation x belongs to the bargaining set $B(\mathcal{E})$ by assumption, the objection $(\bar{S}, \bar{g})$ has a counter-objection $(\bar{T}, z)$, that is, there exists $\{\lambda_i\}_{i\in\bar{T}}$ with $\lambda_i \in (0,1]$ for every $i \in \bar{T}$, such that:

(i) $\sum_{i\in\bar{T}} \lambda_i z_i \leq \sum_{i\in\bar{T}} \lambda_i \omega_i$,

(ii) $U_i(z_i) > U_i(\bar{g}_i)$ for every $i \in \bar{T} \cap \bar{S}$ and

(iii) $U_i(z_i) > U_i(x_i)$ for every $i \in \bar{T} \setminus \bar{S}$.

If $\bar{T} \cap \bar{S} = \emptyset$, then any coalition $T = \bigcup_{i\in\bar{T}} T_i \subset I$, with $\mu(T_i) = \lambda_i$ counter-objects the objection (S, g) via the allocation f_z given by $f_z(t) = z_i$ for every $t \in T_i$.

Otherwise (i.e., $\bar{T} \cap \bar{S} \neq \emptyset$), from the previous condition (ii) we can deduce that for every $i \in \bar{T} \cap \bar{S}$, there exists $A_i \subset S_i$, with $\mu(A_i) > 0$, such that $U_i(z_i) > U_i(g(t))$ for every $t \in A_i$. This is again a consequence of the convexity property of preferences. Let $a = \min\{\mu(A_i), i \in \bar{T} \cap \bar{S}\}$.

Now, take M large enough such that $\alpha_i = \frac{\lambda_i}{M} \leq a$ for every $i \in \bar{T}$. Consider a coalition $T \subset I$ in the continuum economy $\mathcal{E}_c$ with $T = \cup_{i\in\bar{T}} T_i$, such that

- $T_i \subset A_i$, if $i \in \bar{T} \cap \bar{S}$
- $T_i \subset I_i$, if $i \in \bar{T} \setminus \bar{S}$
- $\mu(T_i) = \alpha_i$, for every $i \in \bar{T}$.

Then, defining the step function h as $h(t) = z_i$ if $t \in T_i$, we have that

(i) $\int_T h(t) d\mu(t) = \sum_{i\in\bar{T}} \alpha_i z_i \leq \sum_{i\in\bar{T}} \alpha_i \omega_i = \int_T \omega(t) d\mu(t)$

(ii) $U_i(h(t)) > U_i(g(t))$ for every $t \in T_i$ with $i \in \bar{T} \cap \bar{S}$

(iii) $U_i(h(t)) > U_i(x_i) = U_i(f_x(t))$ for every $t \in T_i$ with $i \in \bar{T} \setminus \bar{S}$

Note that (ii) and (iii) mean $U_t(h(t)) > U_t(g(t))$ for every $t \in T \cap S$ and $U_t(h(t)) > U_t(f_x(t))$ for every $t \in T \setminus S$, respectively. In other words, we have constructed a counter-objection (T, h) for the objection (S, g), and therefore f_x is in the Mas-Colell bargaining set of $\mathcal{E}_c$, which concludes the proof.

Q.E.D.

Proof of Proposition 4.1. Let (S, y) be an objection *à la Aubin* to x. Assume (T, z) is a counter-objection in the sense of Aubin to (S, y). Then, there exist $\lambda_i \in (0, 1]$ for each $i \in T$, such that: $\sum_{i\in T} \lambda_i z_i \leq \sum_{i\in T} \lambda_i \omega_i$; $U_i(z_i) > U_i(y_i)$ for every $i \in T \cap S$ and $U_i(z_i) > U_i(x_i)$ for every $i \in T \setminus S$.

Since (S, y) is a Walrasian objection at prices p we have that $p \cdot z_i > p \cdot \omega_i$, for every $i \in T \cap S$ and $p \cdot z_i > p \cdot \omega_i$, for every $i \in T \setminus S$. This implies $p \cdot \sum_{i\in T} \lambda_i z_i > p \cdot \sum_{i\in T} \lambda_i \omega_i$, which contradicts that z is attainable by T with wights $\lambda_i, i \in T$. Thus, we conclude that (S, y) is a justified objection.

Now, to show the converse, let (S, y) be a justified objection to x and let $a = (a_1, \ldots, a_n)$ be an allocation (not necessarily feasible) such that $a_i = y_i$ if $i \in S$ and $a_i = x_i$ if $i \notin S$.

For every consumer i define $\Gamma_i = \{z \in \mathbb{R}^\ell | U_i(z + \omega_i) \geq U_i(a_i)\} \bigcup \{0\}$ and let Γ be the convex hull of the union of the sets $\Gamma_i, i = 1, \ldots, n$.

Let us show that $\Gamma \bigcap (-\mathbb{R}^\ell_{++})$ is empty. For it, assume that $\delta \in \Gamma \bigcap (-\mathbb{R}^\ell_{++})$. Then, there is $\lambda = (\lambda_i, i = 1, \ldots, n) \in [0, 1]^n$, with $\sum_{i=1}^n \lambda_i = 1$, such that $\delta = \sum_{i=1}^n \lambda_i z_i \in \Gamma$. This implies that the coalition $T = \{j \in \{1, \ldots, n\} \mid \lambda_j > 0\}$ counter-objects (S, y) via the allocation $\hat{z}$ where $\hat{z}_i = z_i + \omega_i - \delta$ for each $i \in T$. Indeed, $\sum_{j\in T} \lambda_j \hat{z}_j = \sum_{j\in T} \lambda_j \omega_j$. Moreover, since $z_i \in \Gamma_i$ for every $i \in T$ and $\delta \ll 0$, by monotonicity of preferences, $U_i(\hat{z}_i) > U(y_i)$ for every $i \in T \cap S$ and $U_i(\hat{z}_i) > U(x_i)$ for every $i \in T \setminus S$. This is a contradiction.

Thus, $\Gamma \bigcap (-\mathbb{R}^\ell_{++}) = \emptyset$, which implies that 0 is a frontier point of Γ. Then, there exists a hyperplane that supports Γ at 0. That is, there exists a price system p such that $p \cdot z \geq 0$ for every $z \in \Gamma$. This means that $p \cdot v \geq p \cdot \omega_i$, if $U_i(v) \geq U_i(a_i)$. Therefore, we conclude that (S, y) is a Walrasian objection.

Q.E.D.

Proof of Lemma 5.1. Let an allocation y be attainable for a coalition S with participation rates $\lambda_i, i \in S$. That is, $\sum_{i\in S} \lambda_i y_i \leq \sum_{i\in S} \lambda_i \omega_i$. It suffices to note that there exists $(\alpha_i, i \in S)$, with $\alpha_i \leq \delta$ for every $i \in S$ such that $\sum_{i\in S} \alpha_i y_i \leq \sum_{i\in S} \alpha_i \omega_i$. For it, take M large enough so that $\alpha_i = \lambda_i / M \leq \delta$, for every $i \in S$. Thus, the same allocation y is also attainable for the same coalition S with participation rates arbitrarily small. The same reasoning holds for the case of both objections and counter-objections.

Q.E.D.

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