

On Sufficiently Diffused Information and Finite-Player Games with Private Information*

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Abstract: This paper takes as its point of departure a 1999 (KRS) example on the non-existence of a pure-strategy Bayesian-Nash equilibrium in one-shot finite-player games with private information, and shows: (i) an equilibrium does exist in the KRS example if the information spaces are upgraded from the unit Lebesgue interval (LI) to the extended LI of Khan-Zhang (2012), (ii) there exists an upgraded KRS-type (counter)example of a game modeled on the extended LI, (iii) if an equilibrium exists in a KRS-type game on an n -fold extended LI, a (counter)example can be constructed on an $(n + 1)$ -fold extended LI, (iv) for any given uncountable compact metric action space, there exists a KRS-type counterexample if information is modeled on spaces identified as satisfying the d-property, (v) the saturation property introduced by Hoover-Keisler (1984) is *sufficient* for the existence of equilibrium in a general class of games, and furthermore, (vi) this property is *necessary* if each player's action set is an uncountable compact metric space. (162 words)

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In commenting on a famous paper which showed nonexistence of equilibrium in an economy with imperfect information, Yaari told the “shocked” economists that the problem lay neither in the solution concept nor in the real world, but rather ... in the model.¹ (Rubinstein 1991)

1 Introduction

In his Walras-Bowley lecture, Rubinstein (1991) discusses the “purification idea” and commenting on the Radner-Rosenthal (1982) purification theorem, observes that the provided conditions are “not mild, and there are simple examples of informational structures by which a mixed-strategy equilibrium could not be purified.”² Radner-Rosenthal themselves introduce their paper with the statement that “if there are any surprises in this paper, we feel they are in the strength of the hypotheses needed to establish existence of pure-strategy equilibria and the ease with which examples may be constructed which do not possess such equilibria.”³ Nevertheless, the Radner-Rosenthal theorem, and the Milgrom-Weber (1985) theorem to follow it,⁴ present a formalization of the following intuition.

It is also sometimes claimed that when information in games is *sufficiently* disparate among the players and when its distribution is *sufficiently* diffuse, the players might as well restrict their attention to pure strategies.⁵

This paper is an exploration of the second use of the qualifier *sufficiently*, and through its presentation of five theorems, resolves a problem that has been open at least since the Radner-Rosenthal-Milgrom-Weber papers written in the early eighties.

The question then is to find language for *sufficient* conditions on the diffusedness of private information under which one would obtain positive results on the existence of pure-strategy Nash equilibrium for a general class of games. A trajectory of partial results that have so far been obtained can be easily laid out.⁶ For games with countable (finite or infinite) action sets, it is

¹See [42, Footnote 18]. The reader may want to note that we join two parts of Rubinstein’s sentences into one.

²See [42, Footnote 7]. Note that there is a minor blemish in this footnote in that its assertion to the contrary, unlike [13], Radner-Rosenthal do not consider a “perturbed game.”

³See the third paragraph of [37, Introduction]. We warn the reader the examples of non-existence of pure-strategy equilibria furnished in [37] concern their independence assumption, and what they refer to as *sufficiently disparate* information. This is not our concern here, and the interested reader may want to look at [4, 38].

⁴See [33, Theorem 4] and the authors’ discussion of how their theorem generalizes the purification results in [37]. For this relationship between the two theorems, also see [19]. It is also worth emphasizing that Milgrom-Weber discuss several issues that go beyond purification of pure-strategy equilibria.

⁵See the first paragraph of [37, Introduction]; the italicization is not in the original. Later on in their text, the authors themselves refer to this sentence in the following terms: “Our opinion is that there is some truth in the imprecise claims mentioned in our first paragraph.”

⁶Our use of the word “partial” is not meant to suggest that the theory of finite, one-shot games of incomplete information, as surveyed the early nineties by Fudenberg-Tirole [11, Chapter 6], has not seen substantial progress in several registers: (i) the existence of mixed-strategy equilibria and their continuity has been shown under rather general conditions in [5, 8], (ii) the approximate and exact purification of such equilibria has been furnished in [4, 38, 22, 25, 8], (iii) the central position of the Dvoretzky-Wald-Wolfowitz (DWW) generalization of Lyapunov’s theorem in the latter results comprehensively delineated in [19, 28, 36, ?], (iv) the direct proof of the existence of pure-strategy equilibria obtained in [29, 10], (v) the importance of cardinality assumptions on the action sets for the theory has been clearly identified in [23, 18, 25], and (vi) the conceptual and technical foundations of the subject clarified, for example, in [3, 47]. Also, under (ii), there is an early paper of Harsanyi’s [13] but its direction is tangential to the work of [37, 4, 33], and therefore to the thrust of this paper; see the discussion of this paper in [42]

sufficient to model diffused information by Lebesgue spaces to obtain a general existence result; see [37, 33, 10] for the case of finite action sets, and [22, 8] for that with countably-infinite ones. For uncountably-infinite action sets, *pace* Fudenberg-Tirole (1991), a relatively early (KRS) counterexample [18] dashed any hopes of Lebesgue spaces being sufficient for such game-theoretic situations.⁷ This example was subsequently circumvented and subdued by the introduction of an atomless Loeb probability space as a model for information in [23, 24, 28, 29],⁸ and thereby furnished one successful formalization of the qualifier *sufficiently* in the Radner-Rosenthal requirement of sufficient diffusedness of information. However, as argued in [23, 24], Loeb spaces are a solution to a variety of ills of idealized limit models in economic theory – asymptotic implementability, permutation-invariance, measurability perspectives – and it is worth asking whether one can isolate the essential property of Loeb spaces that delivers the existence of pure-strategy equilibria of finite games with incomplete information, the sole focus of this paper, without taking the other issues, however important, into its consideration.

In the context of the general theory of large non-anonymous games, this question⁹ has now been definitively resolved by Keisler-Sun [17]. On taking their cue from the *saturation* property formalized in Hoover-Keisler [15] in the form of a saturated filtration,¹⁰ they showed that, loosely speaking, this property that is satisfied by a Loeb measure space is both necessary and sufficient for the existence of Nash equilibria in such games.¹¹ This work has effected a fundamental transformation of the theory by connecting an abstract, and initially somewhat incidental, property of a probability space to the existence of Nash equilibrium of a large non-anonymous game with action sets of uncountable cardinality: the tying together of two literatures with an if-and-only-if result that were previously not tied at all. Furthermore, following [17], once this property was directly connected to Maharam’s [31] classification of measure algebras, progress has been relatively fast and furious. It is now clearly understood that there is at least a triple route to results on saturated spaces. (i) The first route builds on the fact that a Loeb space satisfies the saturation property, and is thereby a saturated space, and, as stressed in [17], results on Loeb spaces can themselves be transferred in a straightforward way to saturated spaces.¹² This is referred to in [30] as the Hoover-Keisler technique. (ii) The second route is to exploit the connection to Maharam’s results: identify the saturation property, again loosely speaking, as the requirement that the restriction of the σ -algebra to any set of positive measure be *not* countably-generated, bypass Loeb spaces and any odour of nonstandard analysis altogether. Such a route, admittedly technically challenging but so far without new substantive consequences from a methodological viewpoint, has been pursued

and in [41].

⁷As pointed out in [18, Footnote 2], there is a general claim by Fudenberg and Tirole of the existence of pure-strategy equilibria in finite games of incomplete information with compact metric action sets in [11, Theorem 6.2 and p. 236]. This led to a corrective of past views that perhaps deserves to be more widely known.

⁸However, under the further specification of information being modeled by atomless Loeb spaces, the existence theory admits a satisfactory generalization to compact metric actions sets, a setting already conceived in Fudenberg-Tirole; also see Footnote 6 for other references.

⁹Without in any way implying it to be a motivation for [17], we observe that this question was posed to the first author by Jean-Francois Mertens at a post-seminar dinner at CORE in May 1996.

¹⁰Their motivation came from the need for a systematic study of the existence of strong solutions for stochastic integral equations; see [6] for a comprehensive discussion.

¹¹Note that this paper was circulated in 2002; see [17, Acknowledgement]. The authors specialized the setting to a single measure space, and henceforth referred to it as a *saturated measure space*.

¹²See also [30, 44], [30, Remarks 2.5 and 2.6] and Footnote 36 below.

by Podczeck to systematically provide direct proofs; see [35, 36]. (iii) A third route, ascribed to Keisler (see [30, Remark 2.6]), remains so far unexplored:

If a property is approximately true in every atomless probability space, then the property is true in every saturated space.¹³

This is how the matter stands for the theory of large anonymous and non-anonymous games.

However, it bears emphasis that despite the fact that both large anonymous and non-anonymous games and finite games with incomplete information involve atomless probability spaces, they admit different interpretations and refer to different substantive registers. Even from a technical point of view they are not isomorphic in any straightforward way.¹⁴ This leads one to ask whether the methodological advances and resulting insights described above apply to finite games with imperfect information? and if so, how? For this, we go back to the drawing board and begin with the KRS example itself. In the first substantive section of the paper, Section 2, we show how this example can be upgraded to a general class of games that we term *KRS-like games*. These are games modeled on compact metric action spaces,¹⁵ on information spaces modeled on arbitrary atomless probability spaces, but with a structure of payoffs that lead their equilibrium distributions to have the same sort of structure as those of the KRS example; see Proposition 1 below. With the KRS example, and KRS-like games in place, we turn in Section 3 to a more detailed consideration the two basic technical underpinnings of the results. The first subsection concerns saturated probability spaces, already referred to above, and the second to what we explicitly identify as “measure spaces satisfying the d-property,” a probability space that does not allow a measurable selection to be chosen from the so-called *d-correspondence* and that induces uniform measure on the range of the correspondence.¹⁶ The prominence that we give to measure spaces having the d-property is, to be sure, new to the literature: it is a point of departure that forms a subtext to the principal results of this entire work.

Next, in Section 4, we turn to the question of how the non-existence of equilibrium in the KRS example can possibly be circumvented, and return to the example itself. This question is motivated by a recent consideration of three correspondences, one of them being the d-correspondence, that do not admit special kinds of measurable selections.¹⁷ It is shown in [26] that in so far as these correspondences are concerned, a simple extension of the Lebesgue interval that goes back to Kakutani [16] in the forties,¹⁸ and one whose σ -algebra is countably generated, is enough to guarantee the requisite selections, and that the resulting techniques have relevance for the theory of large non-anonymous games. However, as emphasized above, large non-anonymous games differ

¹³However, there are results in [20] that can be usefully read from this perspective.

¹⁴See Remark 3 in [32] for an attempted synthesis in the context of distributional strategies, and [8, 9, 10].

¹⁵As we shall see below, this represents a total *volte face* from the Fudenberg-Tirole claim referred to in Footnote 7, and the text it footnotes.

¹⁶This correspondence is reproduced in Figure 2 below, and was referred to in [26] as the *Debreu correspondence* simply as a mnemonic, but as indicated there, Hart-Kohlberg do ascribe it to Debreu in an entirely different context and for an entirely different purpose. Our current use of the letter d for this correspondence, and for the d-property of a measure space based on it, is meant to indicate a situation where each type of agent has a dual best-response. However, if the reader wishes, he or she can capitalize d and make an non-obligatory nod in Debreu’s direction.

¹⁷One of these correspondences derives from the celebrated example of Lyapunov; see Claims 1 to 3 in [26, Section 1].

¹⁸Also see [27]. The difficulty of the work is the need for a translation-invariant extension

technically and substantively from finite games with incomplete information,¹⁹ and two somewhat sharp questions, each apparently standing on their own, naturally suggest themselves. We turn to the first:

- (i) Does this extended probability space, an extension of the Lebesgue interval, resolve the KRS counterexample?

The first subsection of Section 4 answers this question in the affirmative. There exists a pure-strategy Nash equilibrium in the KRS example *if* the information spaces are modeled on the extension of the Lebesgue interval, rather than on the Lebesgue interval itself. And so this appears to be all that there is to it.

Unfortunately this success is more illusory than real. We show in another subsection of Section 4 itself that the KRS example can be modified and resituated on the extended information spaces to yield another troublesome counterexample to the existence of pure-strategy equilibria. This example is of a finite game with information spaces “richer” than those used in the KRS example, but also with the payoffs suitably accommodated and refined to pertain to those spaces. This is a consequence of the well-known facts that

- (a) there exists a continuous onto function from any uncountable compact metric space to $[-1, 1]$,
- (b) there exists a measurable mapping h from an abstract atomless probability space to the usual Lebesgue unit interval such that its induced distribution is the Lebesgue measure itself.²⁰

It is this upgrading of the (counter)example that motivates both a KRS-like game and a measure space satisfying the d-property. But now one can reformulate the question under discussion, and ask:

- (i') Does a further extension of the extended probability space resolve this “new” counterexample?

Perhaps somewhat surprisingly, the answer is again affirmative in that the techniques of [26], and recapitulated in Section 3, are up to the task. However, a recursion now suggests itself and is indeed executable in the form of a general result. Even though a finite game G_n based on an n -fold extension of the Lebesgue interval has no Nash equilibrium, we can construct an $(n + 1)$ -fold extension for which it has an equilibrium. And *none* of these constructed games G_n can have Nash equilibria in any of the sub-extensions. The point is that all these constructed games are KRS-like games with their information spaces satisfying the d-property.²¹ Indeed, this recursive non-existence property culminates in a general theorem; see Theorem 1 below.

¹⁹This point has been made earlier; see Footnote 14 and the text that it footnotes; it bears repetition, however. Also note that the theory of large non-anonymous games has itself been generalized to “non-anonymous games with traits” in [20].

²⁰See, for example, [39] for the first fact, and [17, Lemma 2.1] for the second; also the discussion in Section 2 below.

²¹There is of course a Godelian parallel here. Let T_1 be a suitable theory, which is to say, complete and consistent. Then it admits an undecidable proposition, call it S_1 . Let T_2 be T_1 extended by S_1 , and denoted $T_2 = \{T_1 + G_1\}$. Observe that although G_1 is trivially deducible in T_2 , there is another undecidable in T_2 , say S_2 etc. etc. S_n is never decidable in T_{n-1} . In fact there is a countably-infinite series of pairs of *theories and undecidables*! Extensions of this type never work to furnish a general theory. The authors are grateful to Josh Epstein for bringing the relevance of Godel’s incompleteness theorem to their attention. Josh also singled out parallels to Galois theory whose pursuit in this paper would have taken us too far afield.

The question then is what works? how can this unfortunate recursion be terminated? For this, we turn to the second question mentioned above:

- (ii) Does the existence theory for finite games with private information, as developed for atomless Loeb spaces, generalize to saturated probability spaces?

This leads to an overview of Section 4 below. The clue lies in the “countable extension of a space with a countably-generated σ -algebra.” Returning to [17], we note that a saturated probability space is one whose associated σ -algebra is, loosely speaking, *nowhere countably generated*.²² The fact that in a countably-additive setting, atomlessness, and the consequent *diffusedness* of information, necessitates an information space of uncountable cardinality and a σ -algebra is clear enough,²³ what is a novel in [17] is a turn from the cardinality of the information space to the cardinality of the σ -algebra the space is endowed with. This rather important shift of emphasis deserves to be underlined. On looking at this from another angle, one focused on Harsanyi’s notion of types rather than information, it is clear that the interactive element in a genuine game-theoretic situation leads to a type space of uncountable cardinality even when the original uncertainty of belief involves a finite number of states;²⁴ what is now being emphasized is the uncountable cardinality of the event space rather than the sample space.²⁵ At any rate, for a general existence theorem, one needs to go beyond atomlessness and formalize *sufficiently diffused* by imposing the saturation assumption. Such a theorem is presented in Section 4.2 below.²⁶ The interesting fact is that the recursion is finally closed with a saturated space. In Section 4.3, we establish what Keisler-Sun established for large non-anonymous games: that the existence of a pure-strategy Nash equilibrium implies that the information spaces be *necessarily* saturated.²⁷ Thus, the results furnished in Section 4 bring out, perhaps dramatically, that questions (i), (i’) and (ii) are intimately connected and do not really stand on their own.

Given the above, somewhat multi-layered and sustained, argumentation underlying the results, it might be worthwhile to look at its dialectic from another, and more critical, point of view. Rather than beginning from the KRS example, and its possible resolution, we can turn the matter on its head, and begin instead with the general necessary and sufficient results presented in Section 4. While accepting their if-and-only-if claim – how could one not? – the necessity result can be given a closer and more careful scrutiny. To be sure, the theorem asserts the existence of a game without a pure-strategy Nash equilibrium on any game with incomplete information that is modeled on an atomless probability space whose associated σ -algebra is countably generated, but what of the existing KRS counterexample itself? Of course, the necessity result ensures that for this extended

²²As noted in the third paragraph above, this observation furnishes the second route to saturated spaces, one systematically exploited in [35, 36].

²³See, for example, the discussion in [2].

²⁴See Hart-Heifetz-Samet [14] and the alternative proof in [1].

²⁵Also see the saturated extension of the Lebesgue interval in Sun-Zhang [45].

²⁶Such a result is hinted in [17] but they had a much simpler setting in mind. As the reader will see, the sufficiency result presented below requires some hurdles to be overcome.

²⁷The alert reader may sense a contradiction here to the closing sentence of the previous paragraph, and to Footnote 6. Unlike Gödel’s theorem, we bring a certain completeness and termination to the issue in the form of saturated spaces. However, this may be a consequence of the Polish assumption on the action sets, and one could argue that the introduction of non-separable, albeit compact, action sets would again reactivate the recursion. For some preliminary results in this vein, see [21].

probability space, there will *always* exist a large game without pure-strategy Nash equilibria, but this game may not have any substantive interest. Thus, one can hold the view that as far as the substantive applications are concerned, there is little need for a result that proceeds beyond this extension all the way to a saturated space. This is the point of view articulated in [26] in the context of large non-anonymous games, and further discussion and exploration of whether this is, or is not, only cold comfort for finite games with private information, must be left for future work that turns to concrete applications.²⁸

We now summarize the above narrative with a more succinct road-map for the reader. Section 2 provides a self-contained treatment of the KRS counterexample, and builds on it to define and discuss KRS-like games; Section 3 introduces the saturation and the d-property of probability spaces; and Section 4 shows how an extension of the usual Lebesgue unit interval resolves the counterexample, but brings up a constant and never-ending supply of KRS-like games serving as further counterexamples that need, in turn, further resolution. Section 5 presents the basic “sufficient and necessary” results that involve saturated probability spaces for finite games with private information. As the reader will see from the proofs collected in an Appendix labeled as Section 7, the notions of KRS-like games modeled on a measure space satisfying the d-property proves especially crucial for the necessity part of the theory. The paper concludes with Section 6.

2 KRS-like Games

This section is laid out in two subsections: the first recalls for the reader’s convenience the basic outline of the KRS example [18],²⁹ and the second uses the example to introduce KRS-like games and to characterize the equilibria in such games.

2.1 The KRS Example Revisited

We now review the counterexample in [18]. In this game, there are two players, namely $i = 1, 2$, and each player i can take actions from $A_i = [-1, 1]$, and has the identical private information space $(T_i, \mathcal{T}_i, \mu_i)$ is modeled by the Lebesgue interval³⁰ (I, \mathcal{L}, η) where $I = [0, 1]$. The payoff functions are defined as follows,

$$u_1(a_1, a_2, t) = -|t - |a_1|| + (t - a_1)z(t, a_2), \quad (1)$$

$$u_2(a_1, a_2, t) = -|t - |a_2|| - (t - a_2)z(t, a_1), \quad (2)$$

where $z : [0, 1] \times [-1, 1] \rightarrow \mathbb{R}$ is a function such that for all $t \in [0, 1/2]$,

$$z(t, a) = \begin{cases} a, & \text{if } 0 \leq a \leq t; \\ t, & \text{if } t < a \leq 1; \\ -z(t, -a), & \text{if } a < 0; \end{cases}$$

²⁸See [33, 40] and the references therein, especially to the work of Athey and McAdams. Recall also that the introduction of [33] opens with William Vickrey’s 1961 auction paper.

²⁹Our presentation of this example emphasizes features which the authors were well-aware of more than a decade ago but chose not to make explicit; see Fact 1 below. As such, more than an obligatory reference to the paper is warranted.

³⁰Throughout this paper, we shall reserve $(I = [0, 1], \mathcal{L}, \eta)$ for the unit Lebesgue interval.

and for any $t \in (1/2, 1]$, $z(t, \cdot) = z(1/2, \cdot)$, that is, for all indexes t in $[1/2, 1]$, the functions are identical. See the following Figure 1 for the illustration of $z(t, a)$.

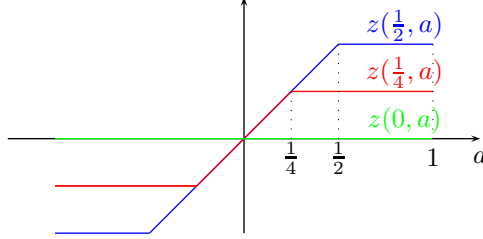


Figure 1: Graph of $z(t, a)$

Note that $u_i > -2$, for all $i = 1, 2$. For simplicity, we will denote this game by

$$\Gamma_0 = \{(T_i, \mathcal{I}_i, \mu_i) = (I, \mathcal{L}, \eta), A_i = [-1, 1], u_i : i = 1, 2\}.$$

Before going to the pure-strategy Nash equilibrium of Γ_0 , we first consider a function which is highly related to the characterization of the equilibrium. Let $w : I \times \mathcal{M}([-1, 1]) \rightarrow \mathbb{R}$ be a function such that for any Borel probability measure ν on $[-1, 1]$, and $t \in I$,

$$w(t, \nu) = \int_A z(t, \cdot) d\nu. \quad (3)$$

For player i , if the distribution of actions of the other player is ν , given that his action is a_i and his private information is t_i , the ex-post expected payoff of player i , denoted by $v_i(a_i, t_i, \nu)$, is

$$\begin{aligned} v_1(a_1, t_1, \nu) &= \int_{a_2 \in A_2} u(a_1, a_2, t_1) d\nu(a_2) = -|t_1 - |a_1|| + (t_1 - a_1)w(t_1, \nu), \\ v_2(a_2, t_2, \nu) &= \int_{a_1 \in A_1} u(a_1, a_2, t_2) d\nu(a_1) = -|t_2 - |a_2|| - (t_2 - a_2)w(t_2, \nu). \end{aligned}$$

That is, each player's ex-post expected payoff is influenced by his opponent's strategy only through the induced distribution of this strategy, and moreover, the influence of this action distribution is only through the function w .

Let (g_1^*, g_2^*) be a pure-strategy Nash equilibrium of Γ_0 , where $g_i^* : I \rightarrow [-1, 1]$ is a Lebesgue measurable function, and assume that the induced distribution $\nu_i^* = \lambda g_i^{*-1}$ on the action set $[-1, 1]$. Then, for η -almost all $t \in I$,

$$\begin{aligned} v_1(g_1^*(t), t, \nu_2^*) &\geq v_1(a_1, t, \nu_2^*) \text{ for all } a_1 \in [-1, 1], \\ v_2(g_2^*(t), t, \nu_1^*) &\geq v_2(a_2, t, \nu_1^*) \text{ for all } a_2 \in [-1, 1]. \end{aligned}$$

Next we are ready to characterize the best response correspondences for this game Γ_0 . From Proposition 2 of [18], it turns out that the function w plays a crucial role in the determination of the best actions. In particular, if $w(t, \nu)$ is zero, either t or $-t$ is the best response if the player's private information is t ; if it is positive, the best response for player 1 is to take t at information t , to take $-t$ at information t for player 2; moreover, the situation is reversed if $w(t, \nu)$ is negative.

We now present some basic facts about the game Γ_0 culminating in an equilibrium non-existence claim.

Fact 1 (1) *The action distributions $\nu_i \in \mathcal{M}([-1, 1])$ for player i , $i = 1, 2$, lead to the following best-response correspondences*

$$B_1(t_1; \nu_2) = \begin{cases} -t_1 \text{ or } t_1, & \text{if } w(t_1, \nu_2) = 0; \\ -t_1, & \text{if } w(t_1, \nu_2) > 0; \\ t_1, & \text{if } w(t_1, \nu_2) < 0. \end{cases} \quad (4)$$

$$B_2(t_2; \nu_1) = \begin{cases} -t_2 \text{ or } t_2, & \text{if } w(t_2, \nu_1) = 0; \\ t_2, & \text{if } w(t_2, \nu_1) > 0; \\ -t_2, & \text{if } w(t_2, \nu_1) < 0. \end{cases} \quad (5)$$

(2) *If ν_1^* and ν_2^* are the induced action distributions of a pure-strategy Nash equilibrium of the game Γ_0 , then $w(t, \nu_i^*) = 0$ for any $t \in I$, and $\nu_i^*([0, a]) = \nu_i^*([-a, 0]) = a/2$ for any $a \in [0, 1/2]$.*

(3) *There does not exist a pure-strategy Nash equilibrium in Γ_0 .*

The first part of Fact 1 is Proposition 2 of [18], and the second part is a combination of Claims 1 and 2 therein. Here is an argument for the third part. If there exists a pure-strategy Nash equilibrium (g_1^*, g_2^*) , then by (2), the induced action distribution for g_i^* should be the uniform distribution on $[-1/2, 1/2]$. Moreover, it follows from (1) that for any player i receiving any private signal $t \in [0, 1]$, his or her best response is either t or $-t$. As a result, restricted to $[0, 1/2]$, g_i^* takes value either t or $-t$ at $t \in [0, 1/2]$ and this induces the uniform distribution on $[-1/2, 1/2]$. However, it is a standard result that there does not exist such a Lebesgue measurable map from $[0, 1/2]$ to $[-1/2, 1/2]$ such that it takes value either t or $-t$ for all $t \in [0, 1/2]$ and yet induces the uniform distribution on $[-1/2, 1/2]$; see the following Figure 2 for an illustration for this standard result; for more details, see Claim 2 of [26].

2.2 KRS-like Games and their Equilibria

The private information spaces in the game Γ_0 are represented by the usual Lebesgue unit interval, the simplest “sufficiently diffused” space. We now turn to the construction of games based on Γ_0 where the private information space of each player is modeled by an arbitrary atomless probability space.

Let $(T_i, \mathcal{T}_i, \mu_i)$, $i = 1, 2$ be two atomless probability spaces, and let $h_i : T_i \rightarrow [0, 1]$ be a \mathcal{T}_i -measurable mapping such that the induced distribution over $[0, 1]$ is the Lebesgue measure η . Consider the following two player private information game to be denoted by Γ_{h_1, h_2} . This game

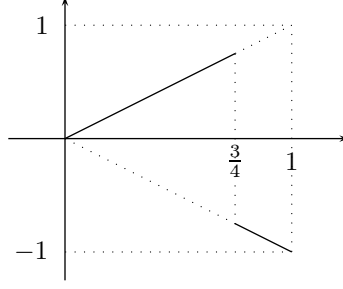


Figure 2: A selection of the d-correspondence that cannot induce a uniform distribution on $[-1, 1]$

is a variation of the above game, Γ_0 , with the same action space $[-1, 1]$, but with $(T_i, \mathcal{T}_i, \mu_i)$ the private information spaces for player i , $i = 1, 2$, and the following payoff functions:

$$\tilde{u}_1(a_1, a_2, t_1) = u_1(a_1, a_2, h_1(t_1)) = -|h_1(t_1) - |a_1|| + [h_1(t_1) - a_1] \cdot z(h_1(t_1), a_2), \quad (6)$$

$$\tilde{u}_2(a_1, a_2, t_2) = u_2(a_1, a_2, h_2(t_2)) = -|h_2(t_2) - |a_2|| - [h_2(t_2) - a_2] \cdot z(h_2(t_2), a_1). \quad (7)$$

In particular,

$$\Gamma_{h_1, h_2} = \{(T_i, \mathcal{T}_i, \mu_i), A_i = [-1, 1], \tilde{u}_i : i = 1, 2\}.$$

In this two-player private information game, the payoff function for player i with private information t_i is the same for this player in the KRS game Γ_0 with private signal $h_i(t_i)$. In other words, in the game Γ_{h_1, h_2} , the payoff function for each player differs from that in Γ_0 only up to a “shift” of the private information. In a rough sense, the game Γ_{h_1, h_2} is akin to a game where each player’s private information space is modeled by the usual Lebesgue unit interval, but it is important to maintain a distinction, and we shall refer to a private information game Γ_{h_1, h_2} as an *KRS-like game* in which the private information spaces $(T_i, \mathcal{T}_i, \mu_i), i = 1, 2$ are taken as given. As a result, the best-response correspondence in the KRS-like game Γ_{h_1, h_2} is based on the best-response correspondence in Γ_0 as follows:

$$B_1(t_1; \nu_2) = \begin{cases} -h_1(t_1) \text{ or } h_1(t_1), & \text{if } w(h_1(t_1), \nu_2) = 0; \\ -h_1(t_1), & \text{if } w(h_1(t_1), \nu_2) > 0; \\ h_1(t_1), & \text{if } w(h_1(t_1), \nu_2) < 0. \end{cases} \quad (4)$$

$$B_2(t_2; \nu_1) = \begin{cases} -h_2(t_2) \text{ or } h_2(t_2), & \text{if } w(h_2(t_2), \nu_1) = 0; \\ h_2(t_2), & \text{if } w(h_2(t_2), \nu_1) > 0; \\ -h_2(t_2), & \text{if } w(h_2(t_2), \nu_1) < 0. \end{cases} \quad (5)$$

Now, suppose that there exists a pure-strategy Nash equilibrium (g_1^*, g_2^*) in the game Γ_{h_1, h_2} , where g_i^* is a function from $(T_i, \mathcal{T}_i, \mu_i)$ to $[-1, 1]$. Let ν_i^* be the induced distribution of g_i^* , i.e., $\nu_i^* = \mu_i \circ (g_i^*)^{-1}$. We next claim that the equilibrium action distribution of each player’s strategy also satisfies the following statement similar to (2) of Fact 1. The proof is provided Section 7.

Proposition 1 *Suppose that ν_1^*, ν_2^* are the induced action distributions of a pure-strategy Nash equilibrium of the game Γ_{h_1, h_2} , then $w(h_i(t_i), \nu_i^*) = 0$ for μ_i -almost all $t_i \in T_i$, and the restricted distribution of ν_i^* on $[-1/2, 1/2]$ is the uniform measure for $i = 1, 2$.*

Proposition 1 is a strengthened version of Fact 1(2). This fact provides the characterization of equilibrium distributions for the KRS game Γ_0 , where the private information space for each player is modeled by the usual Lebesgue unit interval. Proposition 1 offers the characterization of equilibrium distributions for all KRS-like game Γ_{h_1, h_2} when player i 's private information space is fixed to be any atomless probability space $(T_i, \mathcal{T}_i, \mu_i)$; and $h_i : T_i \rightarrow [0, 1]$ is an arbitrary mapping which induces the Lebesgue measure on the interval.

Remark 1 The characterization in Proposition 1 ensures that the distributions (ν_1^*, ν_2^*) induced by a pure-strategy Nash equilibrium (g_1^*, g_2^*) must satisfy the stipulated conditions. Furthermore, the best-response correspondences delineated in Equations (4') and (5'), $g_i^* : T_i \rightarrow [-1, 1]$ takes the value $h_i(t_i)$ or $-h_i(t_i)$ for each t_i . However, for a given atomless probability space $(T_i, \mathcal{T}_i, \mu_i)$ and given mapping $h_i : T_i \rightarrow [0, 1]$, one may not find a \mathcal{T}_i -measurable mapping satisfying the above requirements. We shall discuss this question further in the sequel.

Remark 2 As concluded in [18], the non-existence result in the two-player private information game Γ_0 can be extended to general uncountable action spaces and to multi-player settings by adding dummy players. Similarly, if there does not exist a pure-strategy Nash equilibrium in the KRS-like game Γ_{h_1, h_2} , given the private information spaces, then one can similarly extend such a non-existence result for KRS-like games to general uncountable action spaces and multi-player settings.

3 Basic Technicalities

This section is laid out in two subsections: the first recalls the definition of saturation for the reader's convenience, and the second proposes the notion of a probability space satisfying the *d-property*. The basic intuition for such a property is introduced in Remark 1. We shall follow the following notational conventions. The Lebesgue unit interval has already been specified in Footnote 30 to be $(I = [0, 1], \mathcal{L}, \eta)$. For a Polish (complete separable metric) space X , we denote its Borel σ -algebra by \mathcal{B}_X , and by $\mathcal{M}(X)$ the space of all Borel probability measures associated with the topology of weak convergence. Moreover, probability or measure spaces mean complete countably additive probability or measure spaces. The set of natural numbers is denoted by \mathbb{N} .

3.1 On Saturated Probability Spaces

Given a probability space (V, \mathcal{V}, ν) , for any non-negligible subset $W \in \mathcal{V}$ with $\nu(W) > 0$, denote by $(W, \mathcal{V}^W, \nu^W)$ the probability space restricted to W . Here $\mathcal{V}^W := \{W' \cap W : W' \in \mathcal{V}\}$ and $\nu^W = \nu/\nu(W)$ is the probability measure re-scaled from the restriction of ν to \mathcal{V}^W .

Definition 1 A measure space (V, \mathcal{V}, ν) is called countably generated if there is a countable set $\{W_n \in \mathcal{V} : n \in \mathbb{N}\}$ such that for any $W \in \mathcal{V}$, there is a set W' in the σ -algebra generated by $\{W_n : n \in \mathbb{N}\}$ such that $\nu(W \Delta W') = 0$, where Δ denotes the symmetric difference between W and W' . The measure space is called saturated for any subset $W \in \mathcal{V}$ with $\nu(W) > 0$, the measure space (W, \mathcal{V}^W, ν) is not countably generated.

The usual Lebesgue unit interval is a countably-generated probability space, and therefore not a saturated probability space. In contrast, any atomless Loeb probability space is saturated (see e.g., [15]). For more discussion of the saturated probability spaces, see [15], [17], [26].³¹

We next review some concepts related to the *measure algebra* for a measure space. Let (V, \mathcal{V}, ν) be a measure space. Consider a relation ‘ \sim ’ on the σ -algebra \mathcal{V} as follows. For any two \mathcal{V} -measurable subsets W, W' , define $W \sim W'$ if $\nu(W \Delta W') = 0$. It is clear that ‘ \sim ’ is an equivalence relation on \mathcal{V} . For any $W \in \mathcal{V}$, let \hat{W} be the corresponding equivalence class, i.e., $\hat{W} = \{W' \in \mathcal{V} : W' \sim W\}$. Similarly, denote by $\hat{\mathcal{V}}$ the set of all equivalence classes in \mathcal{V} for the equivalence relation ‘ \sim ’. Define the canonical algebra morphism $\pi_V : \mathcal{V} \rightarrow \hat{\mathcal{V}}$ by letting $\pi_V(W) = \hat{W}$ for any $W \in \mathcal{V}$. It is clear that $\hat{\mathcal{V}}$ is an algebra. Define $\hat{\nu} : \hat{\mathcal{V}} \rightarrow [0, 1]$ by letting $\hat{\nu}(\hat{W}) = \nu(W)$ for any $W \in \mathcal{V}$. The pair $(\hat{\mathcal{V}}, \hat{\nu})$ is said to be the *measure algebra* of the measure space (V, \mathcal{V}, ν) .³²

Definition 2 Let (V, \mathcal{V}, ν) and (V', \mathcal{V}', ν') be two measure spaces, and $(\hat{\mathcal{V}}, \hat{\nu})$ and $(\hat{\mathcal{V}}', \hat{\nu}')$ be their measure algebras respectively. The measure preserving homomorphism $\rho : \hat{\mathcal{V}} \rightarrow \hat{\mathcal{V}}'$, is said to be realized by a measure preserving map $h : V' \rightarrow V$ if for any $W \in \mathcal{V}$, $\pi_{V'}[h^{-1}(W)] = \rho[\pi_V(W)]$, where $\pi_V, \pi_{V'}$ are the canonical morphisms. In other words, the following diagram commutes,

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{h^{-1}} & \mathcal{V}' \\ \pi_V \downarrow & & \downarrow \pi_{V'} \\ \hat{\mathcal{V}} & \xrightarrow{\rho} & \hat{\mathcal{V}}' \end{array}$$

where the homomorphism h^{-1} is naturally derived by h .

We next present a result on atomless countably generated probability spaces (see e.g., Theorem 4.12 of Fremlin [7, p. 937]) that proves instrumental for our results in the sequel.³³ For the Lebesgue interval (I, \mathcal{L}, η) , denote by $(\hat{\mathcal{L}}, \hat{\eta})$ the measure-algebra. Let (V, \mathcal{V}, ν) be an atomless countably-generated probability space associated with the measure algebra $(\hat{\mathcal{V}}, \hat{\nu})$. By Maharam’s theorem (see [31]), there exists a measure-preserving isomorphism $\rho : \hat{\mathcal{L}} \rightarrow \hat{\mathcal{V}}$.

Fact 2 Given an atomless countably generated probability space (V, \mathcal{V}, ν) and the measure algebra isomorphism ρ as above, then ρ can be realized by a measure-preserving mapping h from (V, \mathcal{V}, ν) to the usual Lebesgue unit interval (I, \mathcal{L}, η) .

³¹Note that we are defining the saturation property in a different way from [17, Definition 2.2]. There, a nonatomic probability space (T, \mathcal{T}, μ) is said to have the saturation property for a probability measure ν on the product of Polish spaces $X \times Y$ if for every random variable $f : T \rightarrow X$ which induces the distribution as the marginal measure of ν over X , there is a random variable $g : T \rightarrow Y$ such that the induced distribution of the pair (f, g) on (T, \mathcal{T}, μ) is ν . Furthermore, (T, \mathcal{T}, μ) is said to be saturated if it has the saturation property for every probability measure ν on every product of Polish spaces. It is shown in [15, Corollary 4.5] that these two definitions of the saturation property (for a probability space) are equivalent.

³²See [7] for more on measure algebras.

³³See also [17, Theorem 2.7, p. 1589], [44], [26] for similar applications of this result to the theory of distributions and integral of correspondences.

3.2 On Probability Spaces with the d-Property

For any atomless probability space, (V, \mathcal{V}, ν) , let $\mathcal{L}_{(V, \mathcal{V}, \nu)}$ be the set of measurable functions that induce Lebesgue measure on the usual unit interval. It is well known that $\mathcal{L}_{(V, \mathcal{V}, \nu)}$ is non-empty, see, e.g., [17, Lemma 2.1]. We can now formally define

Definition 3 *An atomless probability space (V, \mathcal{V}, ν) is said to satisfy the d-property if there exists $h \in \mathcal{L}_{(V, \mathcal{V}, \nu)}$ such that the associated d-correspondence $v \mapsto \{h(v), -h(v)\}$ does not have any measurable selection that induces uniform measure on $[-1, 1]$, and the map h is said to be the witness of this d-property.*

We now make a useful observation for a particular witness $h : V \rightarrow [0, 1]$. Given that the d-correspondence $v \mapsto \{h(v), -h(v)\}$ does not have any measurable selection that induces uniform measure on $[-1, 1]$, one can focus on two cases: the nonexistence of the measurable selection takes place either in the first half interval $[0, 1/2]$ or in the second half $[1/2, 1]$. If the second case obtains, one can consider the following $h' : V \rightarrow [0, 1]$,

$$h'(v) = \begin{cases} 2h(v), & \text{if } v \in h^{-1}([0, 1/2]); \\ 2h(v) - 1, & \text{if } v \in h^{-1}([1/2, 1]). \end{cases}$$

It is clear that $h' \in \mathcal{L}_{(V, \mathcal{V}, \nu)}$ and when restricted to $[0, 1/2]$, the associated d-correspondence $t \mapsto \{h'(v), -h'(v)\}$ does not have any measurable selection that induces uniform measure on $[-1/2, 1/2]$. As a result, h' is also a witness of the d-property of (V, \mathcal{V}, ν) . Throughout the paper, we shall therefore assume that a witness h of a space with the d-property is also a witness of this property for the subspace restricted to $h^{-1}([0, 1/2])$.

Finally, the relationship between the d-property and the saturation property can be summarized in the following result. This result plays an essential role in the sequel.

Proposition 2 *(1) A non-saturated probability space (V, \mathcal{V}, ν) has the d-property. (2) In particular, if it is a countably generated probability space, the map h in Fact 2 is a witness of the d-property. (3) A saturated probability space does not have the d-property.*

Remark 3 Parts (1) and (3) of Proposition 2 show that the absence (negation) of saturation, the non-saturation property, is exactly the d-property! Actually, a “baby version” of this idea is already available in the proof of Theorem 3.7 in [17]. Indeed, the authors of [17] present many other characterizations of the saturation property, and our purpose in giving salience to the d-property is that in the context of KRS-like games, it is precisely this characterization of the saturation property that plays an essential role.

4 Finite-Player Games on Lebesgue Extensions

In this section, we turn to the KRS example itself, and ask whether we can obtain a pure-strategy Nash equilibrium in the game Γ_0 by jettisoning the usual Lebesgue unit interval, and simply turning to a more sophisticated atomless probability structure? We will answer this question in the affirmative in Section 4.1 by using the countably-generated extension of the usual Lebesgue unit interval

in [26].³⁴ However, as argued in the introduction, we show in Section 4.2 that this “more sophisticated” and enriched atomless probability space generates its own example of finite-player games without pure-strategy Nash equilibria, but which in turn can be resolved by further enrichment of the σ -algebra. We conclude with a general result formalizing this dialectic in Section 4.3.

4.1 The KRS Example Resolved

In Section 2, we know that in the two-player private information game Γ_0 where the private information structure for each player is the usual Lebesgue unit interval ($I = [0, 1], \mathcal{L}, \eta$), there exists no pure-strategy Nash equilibrium. Let $(I, \mathcal{I}, \lambda)$ be the countably-generated extension of the usual Lebesgue unit interval in Section 2.2 [26].

Next, we consider the following game,

$$\tilde{\Gamma}_0 = \{(T_i, \mathcal{T}_i, \mu_i) = (I, \mathcal{I}, \lambda), A_i = [-1, 1], u_i : i = 1, 2\}.$$

Here $\tilde{\Gamma}_0$ is the same game as Γ_0 except that each player’s private information space is replaced by the countably-generated Lebesgue extension. We now present the following positive result for this new game.

Claim 1 *There exists a pure-strategy Nash equilibrium in the game $\tilde{\Gamma}_0$.*

Compared to Γ_0 , each player in $\tilde{\Gamma}_0$ has a more rich private information space. One natural indication of Claim 1 is that the negative result in game Γ_0 can be resolved if the private information space of each player is updated from the Lebesgue unit interval to its extension. Notice that $(I, \mathcal{I}, \lambda)$ is still a countably generated probability space. One natural question is that if each player’s private information space is modeled by this space, whether we could get a sophisticated theory such that there exists a pure-strategy Nash equilibrium in any private information game. The example below provides a negative answer to this question.

4.2 Yet Another Counterexample

Notice that $(I, \mathcal{I}, \lambda)$ is a countably-generated probability space, by Maharam’s theorem, there is an isomorphism from the measure algebra of the Lebesgue unit interval to that of the Lebesgue extension $(I, \mathcal{I}, \lambda)$. According to Fact 2, this isomorphism can be realized by a measure preserving mapping from $(I, \mathcal{I}, \lambda)$ to (I, \mathcal{L}, η) , denote the mapping by h_0 . It follows from Proposition 2 (2) that h_0 witnesses the d-property for the Lebesgue extension $(I, \mathcal{I}, \lambda)$.

Now consider the following KRS-like game, Γ_{h_0, h_0} .

$$\Gamma_{h_0, h_0} = \{(T_i, \mathcal{T}_i, \mu_i) = (I, \mathcal{I}, \lambda), A_i = [-1, 1], u_i^1 : i = 1, 2\}$$

Now we are ready to present the following negative result. It follows from Proposition 2 (2) and Proposition 1.

Claim 2 *There does not exist a pure-strategy Nash equilibrium in the game Γ_{h_0, h_0} .*

³⁴Note that all the notations, and the meanings as well, of the Lebesgue extensions $(I, \mathcal{I}_n, \lambda_n)$ and the mappings h_n here are the same as those in Section 5.2.3 of [26].

However, as far as the KRS-like game Γ_{h_0, h_0} is concerned, one can resolve this nonexistence result in a similar way as the resolution of the non-existence result for the KRS game Γ_0 as in Section 4.1. In particular, first we can construct a countably generated extension of $(I, \mathcal{I}, \lambda)$, denote it by $(I, \mathcal{I}_1, \lambda_1)$; see Section 5.2.2 of [26] for the construction. Second, we are considering a new game by simply updating each player's private information space from $(I, \mathcal{I}, \lambda)$ to $(I, \mathcal{I}_1, \lambda_1)$,

$$\tilde{\Gamma}_{h_0, h_0} = \{(T_i, \mathcal{T}_i, \mu_i) = (I, \mathcal{I}_1, \lambda_1), A_i = [-1, 1], u_i^1 : i = 1, 2\}.$$

In this way, the nonexistence result in the KRS-like game Γ_{h_0, h_0} can be resolved.

Claim 1' *There does exist a pure-strategy Nash equilibrium in $\tilde{\Gamma}_1$.*

4.3 A General Negative Result

This procedure in Section 4.2 can be continued inductively *at infinitum* in the following sense. First, as in Sections 5.2.2 and 5.2.3 of [26], a sequence of countably-generated probability spaces $\{(I, \mathcal{I}_n, \lambda_n) : n = 0, 1, \dots\}$ can be constructed, where the first countably-generated Lebesgue extension $(I, \mathcal{I}, \lambda)$ is denoted by $(I, \mathcal{I}_0, \lambda_0)$, and for any $n \in \mathbb{N}$, $(I, \mathcal{I}_n, \lambda_n)$ is a countably-generated extension of $(I, \mathcal{I}_{n-1}, \lambda_{n-1})$. Second, if each player's private information space is modeled by $(I, \mathcal{I}_{n-1}, \lambda_{n-1})$, there exists a KRS-like game such that there does not exist a pure-strategy Nash equilibrium. Third, as far as this KRS-like game is concerned, if each player's private information space is modeled by the countably-generated space $(I, \mathcal{I}_n, \lambda_n)$, then the nonexistence result is resolved.

Now we specify the construction of the sequence of KRS-like games. For any $n \in \mathbb{N}$, since $(I, \mathcal{I}_{n-1}, \lambda_{n-1})$ is obtained from $(n-1)$ -times of countably generated extension from the Lebesgue unit interval, it turns out that $(I, \mathcal{I}_{n-1}, \lambda_{n-1})$ itself is a countably-generated probability space. According to Fact 2 and Proposition 2 (2), there is a mapping from $(I, \mathcal{I}_{n-1}, \lambda_{n-1})$ to the Lebesgue unit interval (I, \mathcal{L}, η) , denote the mapping by h_{n-1} , such that h_{n-1} witnesses the d-property for the Lebesgue extension $(I, \mathcal{I}_{n-1}, \lambda_{n-1})$. We consider the KRS-like game $\Gamma_{h_{n-1}, h_{n-1}}$.

$$\Gamma_{h_{n-1}, h_{n-1}} = \{(T_i, \mathcal{T}_i, \mu_i) = (I, \mathcal{I}_{n-1}, \lambda_{n-1}), A_i = [-1, 1], u_i^n : i = 1, 2\},$$

where $u_i^n(a_1, a_2, t) = u_i[a_1, a_2, h_{n-1}(t)]$, $i = 1, 2$ and u_1, u_2 are defined in Equations (1) and (2). Similarly, the associated private information game $\tilde{\Gamma}_{h_{n-1}, h_{n-1}}$ is,

$$\tilde{\Gamma}_{h_{n-1}, h_{n-1}} = \{(T_i, \mathcal{T}_i, \mu_i) = (I, \mathcal{I}_n, \lambda_n), A_i = [-1, 1], u_i^n : i = 1, 2\}.$$

For simplicity, we denote $\Gamma_{h_{n-1}, h_{n-1}}$ by $\Gamma_0 \circ h_{n-1}$.

The arguments above can be illustrated by the following Figure 3.

We are now ready to summarize our argument above in the following theorem, notice that Claims 1 and 2 above are special cases of this result.

Theorem 1 *For each $n \in \mathbb{N}$, there does not exist a pure-strategy Nash equilibrium in the KRS-like game $\Gamma_{h_{n-1}, h_{n-1}}$ but there does exist in game $\tilde{\Gamma}_{h_{n-1}, h_{n-1}}$.*

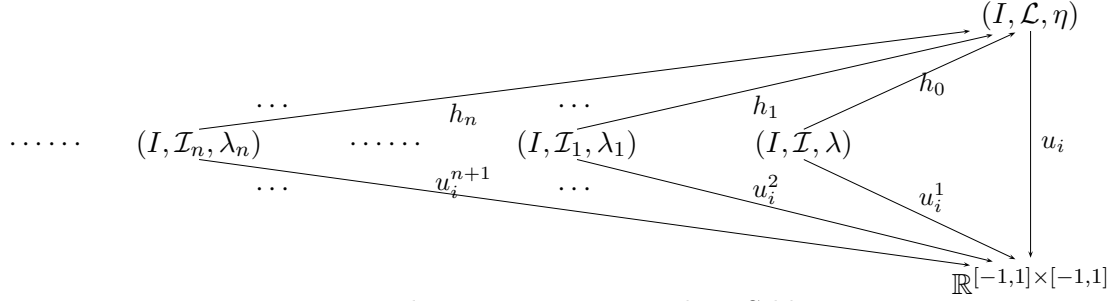


Figure 3: Lebesgue extensions and KRS-like games

We conclude this section by a comment on Lebesgue extensions and private information games. Notice that in the sequence of the Lebesgue extensions $\{(I, \mathcal{I}_n, \lambda_n) : n = 0, 1, \dots\}$, for each $n \in \mathbb{N}$, $(I, \mathcal{I}_n, \lambda_n)$ is a countably-generated extension of $(I, \mathcal{I}_{n-1}, \lambda_{n-1})$. However, no matter how large of n , $(I, \mathcal{I}_n, \lambda_n)$ is still a countably-generated probability space, and thus a space with d-property. This is to say that, if every player's private information space is modeled by this probability space, there always exists a KRS-like game in which there does not exist a pure-strategy Nash equilibrium. Therefore, it is an *impossible* mission for one to establish a general theory of private information games in the sense that if modeling every player's private information space by this probability space, there always exists a pure-strategy Nash equilibrium.

We turn next to further study of issues pertaining to such a general theory.

5 Finite-Player Games on Saturated Spaces

In contrast to the discussion so far, in this section we turn to general results on finite-player games with both public and private information. The section is laid out in three subsections: the first presents the formal model and its underlying assumptions, the second identifies saturated probability spaces as sufficient for the existence of pure-strategy equilibria in such games, and the third establishes the converse. In sum, the three theorems presented below constitute a rather comprehensive and definitive theory.

5.1 Private and Public Information: The Model

A *game with private and public information* Γ consists of a finite set of ℓ players and the following associated spaces and functions. Each player i chooses actions from a compact metric space A_i , and the product $\prod_{j=1}^{\ell} A_j$ is denoted by A . For each player i , a measurable space (T_i, \mathcal{T}_i) represents the private information and events known to the player but not necessarily to other players. A finite or countably infinite set $T_0 = \{t_{0p} : p \in P\}$ represents those states that are to be publicly announced to all the players; denote by \mathcal{T}_0 the power set of T_0 . Another finite or countably infinite set $S_0 = \{s_{0q} : q \in Q\}$ represents the payoff-relevant common states that affect the payoffs of all the players with \mathcal{S}_0 the power set of S_0 . The product measurable space $(\Omega, \mathcal{F}) := (S_0 \times \prod_{j=0}^{\ell} T_j, \mathcal{S}_0 \times \prod_{j=0}^{\ell} \mathcal{T}_j)$ equipped with a probability measure μ constitutes the information space of the game Γ .

Let μ_0 be the marginal probability measure of μ on the countable set $S_0 \times T_0$, and μ_i the

marginal probability measure on (T_i, \mathcal{T}_i) . For each given $t_{0p} \in T_0$ and $s_{0q} \in S_0$, for simplicity, we denote $\mu_0(\{(t_{0p}, s_{0q})\})$ by α_{pq} . Assume that $\alpha_{pq} > 0$ without loss of generality. Let μ^{pq} denote the conditional probability measure of μ on the space $(\prod_{j=1}^{\ell} T_j, \prod_{j=1}^{\ell} \mathcal{T}_j)$ when $t_0 = t_{0p}, s_0 = s_{0q}$; such a conditional probability measure μ^{pq} always exists since both S_0 and T_0 are countable. Furthermore, for each player $i = 1, \dots, \ell$, let μ_i^{pq} be the respective marginal measure of μ^{pq} on the space (T_i, \mathcal{T}_i) .

This conception of information emphasizes a distinction between public and the private, as well as that between payoff and strategy relevant. As such, it provides a blend of the treatments in Radner-Rosenthal and Milgrom-Weber, and has been emphasized in recent work of Fu; see [8, 9, 10]. We shall need the following assumptions on the information structure, $(\Omega, \mathcal{F}, \mu)$: the first concerns *dispersedness* and the second *diffusedness*, both emphasized in the pioneering papers on the subject discussed in the introduction.

Assumption 1 *Given the public and payoff-relevant common information structure, $T_0 = \{t_{0p} : p \in P\}$, $S_0 = \{s_{0q} : q \in Q\}$, for each $p \in P, q \in Q$, the players' private information is conditionally independent if*

$$\mu^{pq} = \prod_{i=1}^{\ell} \mu_i^{pq}. \quad (8)$$

Assumption 2 *The players' strategy-relevant private information is diffused if the marginal measure of μ on (T_i, \mathcal{T}_i) , μ_i , is atomless for each player i .*

Next we turn to the payoffs. For each player i , his payoff function $u_i : A \times S_0 \times T_i \rightarrow \mathbb{R}$ depends on the actions chosen by all the players, a payoff relevant common state $s_0 \in S_0$, and on his own private information $t_i \in T_i$. We consider the following assumptions on it, principal among which is the equicontinuity assumption made in

Assumption 3 *For each player i , (i) $u_i(\cdot, s_0, t_i)$ is a continuous function on A when s_0 and t_i are fixed; (ii) for each $a \in A$ and $s_0 \in S_0$, $u_i(a, s_0, \cdot)$ is \mathcal{T}_i -measurable on T_i ; and (iii) there is an integrable function ϕ_i on $(\Omega, \mathcal{F}, \mu)$ such that for each payoff function u_i , $|u_i(a, s_0, t_i)| \leq \phi_i(s_0, t_0, t_1, \dots, t_{\ell})$ holds for each $a \in A$, and each $(s_0, t_0, t_1, \dots, t_{\ell}) \in \Omega$.*

In what follows, when i is given, we shall abbreviate a product over all indices $1 \leq j \leq \ell$ except for $j = i$ by $\prod_{j \neq i}$; i.e., $\prod_{j \neq i}$ means $\prod_{1 \leq j \leq \ell, j \neq i}$. For each player $i = 1, \dots, \ell$, we shall use the following (conventional) notation: $A_{-i} = \prod_{j \neq i} A_j$. For any action profile $a \in A = \prod_{j=1}^{\ell} A_j$, we write a_{-i} for the projection of a into A_{-i} , as a result a can be written as (a_i, a_{-i}) .

We now present the concepts of the strategies. A *mixed strategy* for player i is a measurable mapping from her information space $(T_0 \times T_i, \mathcal{T}_0 \otimes \mathcal{T}_i)$ to $\mathcal{M}(A_i)$, the space of all Borel probability measures on the Polish space, A_i . A *pure strategy* is an $\mathcal{T}_0 \otimes \mathcal{T}_i$ -measurable mapping from $T_0 \times T_i$ to A_i , and it can be regarded as a mixed strategy using Dirac measures. A *mixed (pure) strategy profile* $f = (f_1, \dots, f_{\ell})$ is a tuple of mixed (pure) strategies, in which f_i specifies a mixed (pure) strategy for player i . In the sequel, for each player i and his mixed strategy f_i , then for any t_0, t_i , let $f_i(t_0, t_i; da_i)$ denote the integration operator with respect to the Borel probability measure $f_i(t_0, t_i)$. Therefore, given any mixed strategy profile $f = (f_1, \dots, f_{\ell})$, the expected payoff for player i is

$$U_i(f) := \int_{\Omega} \int_A u_i(a, s_0, t_i) f_1(t_0, t_1; da_1) \cdots f_{\ell}(t_0, t_{\ell}; da_{\ell}) d\mu(\omega), \quad (9)$$

where for each $t_0 \in T_0$ and $t_i \in T_i$, the inner integral on A is the iterated integral on A_ℓ, \dots, A_1 respectively. A mixed strategy profile $f = (f_i, f_{-i})$ is called a *Nash equilibrium* for the game Γ if for every player i , $U_i(f_i, f_{-i}) \geq U_i(f'_i, f_{-i})$ for any mixed strategy f'_i of player i .

5.2 A Sufficiency Result

Next we turn to the question of the existence of a pure-strategy equilibrium in the model presented above. There have been two canonical ways to approach this: (i) an *indirect* method which shows the existence of a mixed-strategy equilibrium, and then proceeds to show that each such equilibrium can be “purified, (ii) a *direct* method which makes no reference to mixed-strategies, and unlike (i), relies on an appropriate fixed-point theorem. The first approach was undertaken in the pioneering papers of Radner-Rosenthal and Milgrom-Weber,³⁵ and since [19], shown to revolve around the 1951 Dvoretzky-Wald-Wolfowitz theorem. Under the rubric of the purification of measure-valued maps, it has been considerably generalized by Podczeck, Loeb-Sun and Wang-Zhang.³⁶ The second approach is due to Khan-Sun [22, 24] and does revolve around the Fan-Glicksberg fixed-point theorem applied to a suitable space of distributions, and therefore relies on a theory of the distribution (or law) of a set-valued random variable.³⁷

However there is an added aspect to the theory. Radner-Rosenthal distinguished between payoff-relevant and strategy-relevant information, while Milgrom-Weber differentiated between public and private information, and recent work of Fu and others has considered a synthetic treatment that incorporates both aspects of the model; see [8, 9, 10]. These distinctions necessitated an extension of the theory of distributions from a scalar to a vector-valued measure for the direct approach: for finite action sets in [10], and for uncountable action sets, but with information modeled as Loeb spaces, in [29].³⁸ We show the existence of pure-strategy equilibria through the indirect approach, and rely on the following existence theorem for a mixed-strategy equilibrium that derives from [8, Theorem 1].

Lemma 1 *Supposed that Assumptions 1, 2 and 3 hold, then there exists a mixed strategy equilibrium for the game Γ .*

Remark 4 As is well-known, Milgrom-Weber appealed to absolutely continuous information and to equicontinuous payoffs to establish their existence theorem on distributional strategies.³⁹ This

³⁵See [33, Theorem 1], and the generalization offered in [5, Theorem 3.1]. Note that these results make no assumptions, as they ought not, to restrict the cardinality of the action sets.

³⁶See [36], [28, 30]; the first and the third revolve around saturated spaces. The point of [30] is to show that the results in [28] can be transferred from a Loeb to a saturated setting in a straightforward way, what we referred to as the Hoover-Keisler approach in the introduction; see Footnote 12 and the text it footnotes. In [46], it is shown that there may be an infinity of purifications of a mixed-strategy equilibrium.

³⁷Such a theory for countably infinite action sets is presented in [22] as a direct consequence of the Bollobás-Varopoulos extension of the marriage lemma; and for uncountably infinite sets in [24] as a consequence of a more technically-demanding theory, also relying on the Bollobás-Varopoulos theorem, in [43].

³⁸Indeed, the relevant mathematical framework in [29] is to establish the required properties for the distribution of correspondences via countably infinite-dimensional vector measures, whereas [10] need only to consider finite-dimensional vectors measures.

³⁹In particular, the marginal probability measure of μ on $(\prod_{i=1}^\ell T_i, \otimes_{i=1}^\ell \mathcal{T}_i)$, denoted by μ' , is absolutely continuous with respect to $\prod_{i=1}^\ell \mu_i$, where μ_i is the marginal probability measure of μ on (T_i, \mathcal{T}_i) . Here, given two measures μ

result is generalized in [5] by replacing the equicontinuous condition by a condition similar to our Assumption 3. Fu [8] proved the existence of mixed strategy equilibrium by applying Theorem 3.1 of [5]. We prove Lemma 1 as a consequence of [8, Theorem 1], but we need to show that Assumption 1 implies the absolute continuity condition. This is non-trivial and is relegated to Section 7.

Next, we turn to the issue of purification of this mixed-strategy equilibrium. Following [19], we define the concept of strong purification for our model.

Definition 4 *A pure-strategy profile $g = (g_1, \dots, g_\ell)$ is said to be a strong purification of a mixed strategy profile $f = (f_1, \dots, f_\ell) = (f_i, f_{-i})$ if the following four conditions are satisfied for each player i .*

1. $U_i(f) = U_i(g)$.
2. For any given mixed strategy \tilde{f}_i of player i , $U_i(\tilde{f}_i, f_{-i}) = U_i(\tilde{f}_i, g_{-i})$.
3. For each $p \in P$ and $q \in Q$, given $t_0 = t_{0p}$, $s_0 = s_{0q}$, $g_i(t_{0p}, \cdot)$ and $f_i(t_{0p}, \cdot)$ have the same conditional distribution on A_n , i.e., $\int_{T_i} f_i(t_{0p}, t_i; \cdot) d\mu_i^{pq}(t_i) = \mu_i^{pq} g_i^{-1}(t_{0p}, \cdot)$.
4. For all p and $q \in Q$, $g_i(t_{0p}, t_i) \in \text{supp } f_i(t_{0p}, t_i)$ for μ_i^{pq} -almost all $t_i \in T_i$.

We can now apply the purification principle based on saturated probability spaces, as in [30, Corollary 2.3], to purify the mixed-strategy Nash equilibrium established in Lemma 1 to obtain a pure-strategy Nash equilibrium. In particular, items 1 and 2 in Definition 4 guarantee that if the mixed strategy f is a Nash equilibrium, so is its strong purification, thus furnishing a pure-strategy Nash equilibrium. The result is the following.

Theorem 2 *Suppose that Assumptions 1-3 hold, and for each player i , the marginal private information space $(T_i, \mathcal{T}_i, \mu_i)$ is a saturated probability space, then there exists a strong purification for every mixed strategy in Γ . Moreover, there exists a pure-strategy Nash equilibrium in the game Γ .*

Theorem 2 contributes to the literature in two different ways. On the one hand, it generalizes the results in [29] to finite-player games with countable public information states, uncountable infinite action sets and the private information spaces are modeled by saturated probability spaces, in which we work with the standard measure-theoretic framework.⁴⁰ On the other hand, it generalizes the “purification principle” in [28] from a Loeb space to any saturated space. We now conclude this section with the following remarks.

and λ on the measurable space (T, \mathcal{T}) , μ is said to be *absolutely continuous* with respect to λ if for any $S \in \mathcal{T}$ with $\lambda(S) = 0$, then $\mu(S) = 0$.

⁴⁰We note that in the context of the sufficiency theory, [17, Footnote 1] write: “In fact, the full statements of Theorem 1 for large games and Theorem 3 for finite-player games with incomplete information in [24] can be restated on saturated probability spaces instead of Loeb spaces; exactly the same proofs work in this more general situation.” The point is that we present our sufficiency result in a richer setting proposed and studied in [8, 9, 10, 29], and is technically more challenging than the one considered in [24].

Remark 5 There is little doubt that on using the techniques in [19], and supplementing then with the arguments developed here, Theorem 2 can be generalized to a setting pursued in [8] where each player’s private information is divided into payoff-relevant private information and strategy-relevant parts. We leave this to the interested reader.

Remark 6 We leave the question of a direct proof of Theorem 2 as an open problem of methodological interest.

5.3 A Necessity Result

Next we turn to the necessity results, the backbone of the theory that is developed and reported in this paper. This asks what the existence of pure-strategy Nash equilibrium of each game with private and public information chosen from a class of games implies for the space of information on which it is modeled. In particular, given a formalization of information spaces under Assumptions 1 and 2, the assurance of a pure-strategy Nash equilibrium for a class of games whose payoff structures satisfy Assumption 3, the marginal probability space of the information structure conditional on any pair of public and common states *must* be saturated. We present two results, the first pertaining to a simple setting where the sets P and Q of public information states are singletons. Such a result serves as an essential input into the proof of the result for the general case.

Theorem 3 *Given a finite set of ℓ players, for each player i , suppose that A_i is a uncountable compact metric space. Let $(\Omega, \mathcal{F}, \mu) = (\prod_{i=1}^{\ell} T_i, \otimes_{i=1}^{\ell} \mathcal{T}_i, \prod_{i=1}^{\ell} \mu_i)$ be the information structure.⁴¹ If there exists a pure-strategy Nash equilibrium in any private information game satisfying Assumptions 2 and 3, then $(T_i, \mathcal{T}_i, \mu_i)$ must be a saturated probability space for every player i .*

Finally, as implicitly brought out in its proof presented in Section 7, the necessity result in Theorem 3 can be re-phrased in the following more powerful way.

Theorem 3' *If there exists a pure-strategy Nash equilibrium in every KRS-like game, then the private information space for each player must be saturated probability space.*

This is a more powerful result in the sense that it derives the necessity of saturation from a hypothesis pertaining to what is apparently a limited class of games of private information.

Next we turn to a more general setting where the set of public states, as well as the common payoff relevant states, can be of countable cardinality. We have the following result.

Theorem 4 *Given a finite set of ℓ players, given two countable sets S_0 and T_0 , for each player i , suppose that A_i is a uncountable compact metric space. Let $(\Omega, \mathcal{F}, \mu)$ be the private information structure satisfying Assumptions 1 and 2. If any private information game satisfying Assumption 3 has a pure-strategy equilibrium, then $(T_i, \mathcal{T}_i, \mu_i^{PQ})$ is a saturated probability space for every player i , for any $t_{0p} \in T_0$ and $s_{0q} \in S_0$.*

Remark 7 It is to be noted that the negative results stated in Section 2, and culminating in Proposition 2 and Theorem 1, can be viewed as special cases of the above Theorems 3 and 4.

⁴¹Notice that the conditional independence mentioned in Assumption 1 is trivially satisfied.

Indeed, in these negative results, a player’s private information is modeled either by the Lebesgue unit interval or some countably-generated extension of the Lebesgue interval, and as such is not saturated probability space.

The proofs of Theorem 4 is furnished in Section 7 below. Note that Theorem 3 can be viewed as a special case of Theorem 4 simply by taking the index sets P, Q to be two singletons. The key idea is that in a finite-player game with private information and public information, when the pair of public information states (t_{0p}, s_{0q}) is announced, then by Assumption 1, conditional on such an event, the game is reduced to a private information game with private information structure $(\Pi_{i=1}^{\ell} T_i, \otimes_{i=1}^{\ell} \mathcal{T}_i, \Pi_{i=1}^{\ell} \mu_i^{pq})$. If there exists a pure-strategy Nash equilibrium in each game with private information and public information, there will also exist a pure-strategy Nash equilibrium in each such private information game for any fixed (t_{0p}, s_{0q}) .

6 Conclusion

In conclusion, we provide a comprehensive investigation of the fundamental insight of Radner-Rosenthal [37] that players will restrict their attention to pure strategies when information in games with large action spaces is sufficiently diffused. If each player’s private information is modeled by saturated probability spaces (which is to say, spaces not satisfying the d-property), Theorem 2 shows that there always exists a pure-strategy Nash equilibrium in a broad class of games. This is underscored by the finding in [46] that there are *many* pure-strategy Nash equilibria corresponding to a single mixed-strategy equilibrium. As a result, the players have considerable incentive to restrict their attention to pure strategies. On the other hand, Theorems 3 and 4 make this modeling of private information by saturated probability spaces (or again, spaces not satisfying the d-property) more specific. If any player’s private information is modeled by a non-saturated probability space, there is a game situation where there does not exist any pure-strategy Nash equilibrium, and as a result, the question of the players restricting their attention to pure strategies does not arise.

The ideas behind the results can perhaps be put another way. For any game situation falling within the class of games studied in this paper, and without a pure-strategy Nash equilibrium, it must be the case that some player’s private information was modeled by a non-saturated probability space. And as in Section 4, there is a more informative structure, an extension of the non-saturated probability space, to overcome the negative result. However, as illustrated in Section 4, this overcoming pertains only to the particular game that is being considered, and the hope for a general theory for the existence of pure-strategy Nash equilibrium in this way proves fatuous. There is always a further counterexample in which there does not exist a pure-strategy Nash equilibrium. The underlying reason for this is simply that a non-saturated probability space, when extended by a non-saturated one,⁴² remains a non-saturated space: there is simply no escape from the abyss.

⁴²We may recall here the result of [45] in which the Lebesgue interval, when extended by a Loeb measure space, yields a Loeb measure space.

7 Appendix: Proofs of Results

We present four theorems in this paper, and without necessarily privileging one over the other, let us note that from a technical point of view, the most difficult, and perhaps subtle, part of the theory concerns the necessity results, Theorems 3 and 4. The latter is proved through a construction that allows its hypotheses to be reduced to those of Theorem 3, and thereby invoke its conclusion. And so we begin this Appendix on the proofs of the results by alerting the reader to the fact that Proposition 1 pertaining to KRS-like games, and Proposition 2 to measure spaces satisfying the d-property, while of interest in their own right, feed directly into the proof of Theorem 3, as well in the proof of Theorem 1.

We begin with Proposition 1. As indicated above, this proposition establishes that the structure of the equilibrium of the KRS example persists when we turn to the more general KRS-like game. This involves a rather tedious but essential verification of the argumentation presented in [18] for the generalized context.

Proof of Proposition 1 The claims constituting Proposition 1 are a variation (analogues) of the combination of Claims 1 and 2 in [18]. In [18], Claim 2 is implied by Claim 1 and the proof of Claim 1 consists of a series of Lemmas and Propositions. Proposition 1 can be proved in a similar way. Since the results in [18] were established on the usual Lebesgue unit intervals and for the game Γ_0 , and Proposition 1 is on $(T_i, \mathcal{T}_i, \mu_i)$, which is mapped to the usual Lebesgue interval via h_i , and the KRS-like game Γ_{h_1, h_2} , a series of results in [18] need to be modified and checked for the changed context. For this checking, we adopt the following conventions. First, we do not provide a proof of any claim that involves re-writing from [18] with new but obvious notation. Second, we reference the corresponding result in [18] by italics, and use \square instead of \blacksquare when the argument is complete.

[Lemma 1] *If (f_1^*, f_2^*) is an equilibrium then $(-f_1^*, -f_2^*)$ is also an equilibrium.*

Proof: The proof is the same as [Lemma 1]. The only difference here is the need to replace $\lambda \circ f_i^{*-1}$ by $\mu_i \circ f_i^{*-1}$, since f_i^* is a map from T_i to $[-1, 1]$. \square

[Proposition 2]. *The best response in the game $\tilde{\Gamma}$ is made clear in Equations (4') and (5').*

[Lemma 2] *Let (f_1^*, f_2^*) be an equilibrium of the game. Then, for any $i = 1, 2$, and any $t_i \in h_i^{-1}([1/2, 1])$, $w(h_i(t_i), \nu_i^*) = 0$, where $\nu_i^* = \mu_i \circ f_i^{*-1}$ on the action set $[-1, 1]$.*

Let $d(t, \nu) \equiv \nu([t, 1]) - \nu([-1, -t])$, for any $t \in [0, 1]$ and $\nu \in \mathcal{M}([-1, 1])$.

[Proposition 3] Let F be a measurable subset of T_i . Define a \mathcal{T}_i -measurable function f from T_i to $[-1, 1]$ by $f(t_i) = h_i(t_i)$ if $t_i \in F$ and $f(t_i) = -h_i(t_i)$ if $t_i \notin F$. Let ν be the induced measure $\mu_i \circ f^{-1}$. For any $r \in (0, 1/2)$, $w(r, \nu)$ is differentiable with its derivative equal to $d(r, \nu)$. Furthermore, $w'_+(0, \nu) = d(0, \nu)$ and $w'_-(1/2, \nu) = d(1/2, \nu)$ where $w'_+(0, \nu)$ and $w'_-(1/2, \nu)$ respectively denote the right and left derivatives of w at 0 and 1/2.

Proof: Since h_i induces the Lebesgue measure on $[0, 1]$, as a result, the induced distribution of f satisfies the following properties, $\nu([-t, 0]) + \nu([0, t]) = \nu([-t, t]) = t$ for any $t \in [0, 1]$. The proof of this result is the same as that in [18]. \square

[Proposition 4] Let F, f , and ν be as in Proposition 3, and $0 \leq r < t \leq (1/2)$. If the open interval

$(r, t) \subseteq h_i(F)$, then $w(t, \nu) - w(r, \nu) = (t - r)[d(r, \nu) - (1/2)(t - r)]$. If $(r, t) \cap h_i(F) = \emptyset$, then $w(t, \nu) - w(r, \nu) = (t - r)[d(r, \nu) + (1/2)(t - r)]$.

Proof: If $(r, t) \subseteq h_i(F)$, $h_i^{-1}((r, t)) \subset F$, then $\nu((r, t)) = \mu_i \circ f^{-1}((r, t)) = \mu_i \circ h_i^{-1}((r, t)) = t - r$. As a result, $\nu((-t, -r)) = 0$. The other parts are the same as in [18]. \square

We now alert the reader to a more substantial departure from the argumentation in [18]. There, the idea of a ‘‘permissible interval’’ is defined in terms of players, and we transfer it to equilibrium distributions ν_i^* on $[-1, 1]$. Accordingly, we shall say that a closed interval $[r, s]$, $r \neq s$ in $[0, 1/2]$ is *permissible w.r.t.*, ν_1^* if $w(r, \nu_1^*) = w(s, \nu_1^*) = 0$ and $w(t, \nu_1^*) \neq 0$ and does not change sign for all $t \in (r, s)$. Two permissible intervals $[r, s]$ and $[a, b]$ are said to be *adjacent* if they have a common endpoint, i.e, either $s = a$ or, $r = b$. Two permissible intervals $[r, s]$ and $[a, b]$ are said to be *opposite in sign* if $w(t, \nu_1^*)w(t', \nu_1^*) < 0$ for all $t \in (r, s)$ and for all $t' \in (a, b)$. Finally, an interval $[r, s]$ is said to be *larger* than another interval $[a, b]$ if $s - r > b - a$. The following results are now routine.

[Lemma 3] Let $[r, s]$ be a positive permissible interval w.r.t. ν_1^* . Then $d(r, \nu_1^*) \geq 0$ and $d(s, \nu_1^*) \leq 0$. Furthermore, (i) $w(r, \nu_2^*) < 0 \Rightarrow d(r, \nu_1^*) > 0$; (ii) $w(s, \nu_2^*) < 0 \Rightarrow d(s, \nu_1^*) < 0$.

[Lemma 4] Let $[r, s]$ be a positive permissible interval w.r.t. ν_1^* . (i) $d(r, \nu_2^*) \geq (s - r)/2 \Rightarrow w(r, \nu_2^*) < 0$. (ii) $d(s, \nu_2^*) \leq -(s - r)/2 \Rightarrow w(s, \nu_2^*) < 0$.

[Lemma 5] Every permissible interval w.r.t. ν_i^* has a larger adjacent permissible interval opposite in sign.

We are now ready to prove Proposition 1. For the first statement, the proof is the same as that in [18]. For the second part where the restricted distribution of ν_i^* on $[-1/2, 1/2]$ is the uniform measure, as in [18], it follows from the first part and from [Proposition 3]. \blacksquare

Proof of Proposition 2. We first prove (2) where (V, \mathcal{V}, ν) is a countably-generated probability space. Let $h : V \rightarrow [0, 1]$ be the map in Fact 7 where certain isomorphism between the measure algebra of the Lebesgue unit interval and the measure algebra (V, \mathcal{V}, ν) is realized by h .

We next show by contradiction that there does not exist a \mathcal{V} -measurable selection of the d-correspondence $v \rightsquigarrow \{h(v), -h(v)\}$ such that the induced distribution is the uniform measure on $[-1, 1]$. Suppose not, let $g : V \rightarrow [-1, 1]$ be such a selection of the d-correspondence. Let $W = \{v \in V : g(v) = h(v)\}$ and it is clear that $W \in \mathcal{V}$. Since the map h induces an isomorphism between the measure algebras of the Lebesgue unit interval and the measure algebra (V, \mathcal{V}, ν) , by [26, Corollary 1], there exists a \mathcal{L} -measurable subset S of $[0, 1]$ with $\nu[W \Delta h^{-1}(S)] = 0$.

Now define $\psi : [0, 1] \rightarrow [-1, 1]$ by letting $\psi(t) = t$, if $t \in S$ and $-t$ if $t \in [0, 1] \setminus S$. By definition, ψ is a \mathcal{L} -measurable mapping. For any $a \in [0, 1]$, we next show that $\eta\psi^{-1}(0, a) = \eta\psi^{-1}(-a, 0) = a/2$. Indeed we can verify the the first part $\forall a \in [0, 1]$ as follows:

$$\begin{aligned} \eta\psi^{-1}(0, a) &= \eta((0, a) \cap S) = \nu[h^{-1}((0, a) \cap S)] && \text{(since } \nu h^{-1} = \eta) \\ &= \nu[h^{-1}(0, a) \cap h^{-1}(S)] = \nu[h^{-1}(0, a) \cap W] && \text{(since } \nu[W \Delta h^{-1}(S)] = 0) \\ &= \nu g^{-1}(0, a) = a/2. && \text{(as a result of the definitions of } S \text{ and } g) \end{aligned}$$

Similarly, we can show that $\eta\psi^{-1}(-a, 0) = a/2$ for any $a \in [0, 1]$. As a result, ψ is a Lebesgue measurable mapping whose induced distribution is uniform in $[-1, 1]$. This contradicts [26, Claim 2].

When restricted to $h^{-1}([0, 1/2])$, the subspace of (V, \mathcal{V}, ν) is still a countably-generated measure space, and we can show that the restricted d-correspondence does not have measurable selection which induces the uniform measure on $[-1/2, 1/2]$. Therefore, the countably-generated probability space (V, \mathcal{V}, ν) has the d-property, and the map h is a witness of the d-property.

We next prove Part (1) where (V, \mathcal{V}, ν) is not a saturated probability space. Since (V, \mathcal{V}, ν) is not a saturated probability space, it follows from Definition 1 that there exists a \mathcal{V} -measurable subset $W \subseteq V$ with $\nu(W) > 0$ (denoted $\nu(W)$ by s) such that the restricted measure space (W, \mathcal{V}^W, ν) is countably-generated, where \mathcal{V}^W is the σ -algebra of \mathcal{V} restricted to W . As a result of Maharam's theorem (see [31]), the measure algebra of $([0, s], \mathcal{L}^s, \eta)$ is isomorphic to the measure algebra of (W, \mathcal{V}^W, ν) , where \mathcal{L}^s is the σ -algebra of all Lebesgue sets on $[0, s]$. According to Fact , this isomorphism can be realized by a measure preserving map h_W from (W, \mathcal{V}^W, ν) to $([0, s], \mathcal{L}^s, \eta)$. We now consider the measure space restricted to $V \setminus W$, the complementary set for W in V . Since it is also an atomless measure space, by [17, Lemma 2.1] again, there exists a measurable map $h_{V \setminus W} : V \setminus W \rightarrow [s, 1]$ such that the induced distribution of $h_{V \setminus W}$ on $[s, 1]$ is the Lebesgue measure on $[s, 1]$. Let h_1 be a map from (V, \mathcal{V}, ν) to the Lebesgue interval defined as follows,

$$h(v) = \begin{cases} h_W(v), & \text{if } v \in W; \\ h_{V \setminus W}(v), & \text{if } v \in V \setminus W. \end{cases} \quad (10)$$

It is clear that h is a \mathcal{V} -measurable map and the induced distribution of h is the Lebesgue measure η , i.e., $h \in \mathcal{L}_{(V, \mathcal{V}, \nu)}$.

We next show that the map h constructed as above is a witness of the d-property for the probability space (V, \mathcal{V}, ν) . That is, there does not exist a \mathcal{V} -measurable selection of the associated d-correspondence, $v \rightarrow \{h(v), -h(v)\}$, for all $v \in V$ such that the induced distribution of this selection is uniform on $[-1, 1]$. Suppose not, consider the restriction of the d-correspondence to W , there is a measurable selection of the restricted correspondence such that this selection induces the uniform measure on $[-s, s]$. It is a contradiction to the assertion of Part (2). Similarly, there does not exist a measurable selection of the correspondence, when restricted to $h^{-1}([0, 1/2])$, such that this selection induces the uniform measure on $[-1/2, 1/2]$. Therefore, the non-saturated probability space (V, \mathcal{V}, ν) has the d-property, and the map h is a witness of the d-property.

Finally, we provide the proof of Part (3). Let h be an arbitrary map from the saturated probability space (V, \mathcal{V}, ν) to the Lebesgue unit interval where the induced distribution of h is uniform on the interval. Consider the d-correspondence, denote it by Ψ , $\Psi(v) = \{h(v), -h(v)\}$ for all $v \in V$. Let \mathcal{D}_Ψ be the set of induced distributions of the \mathcal{V} -measurable selections of Ψ . Namely, both h and $-h$ are selections of Ψ and they induce the uniform distribution on $[0, 1]$ and $[-1, 0]$ respectively. By [17, Theorem 3.6, P(2)], \mathcal{D}_Ψ is a convex set. As a result, the uniform measure on $[-1, 1]$, which is equal to the sum of half the uniform measure on $[-1, 0]$ and half the uniform measure on $[0, 1]$, is an element of \mathcal{D}_Ψ . Hence, there exists a selection of Ψ which induces the uniform distribution of $[-1, 1]$. Therefore, any map $h \in \mathcal{L}_{(V, \mathcal{V}, \nu)}$ can not be a witness of the d-property, that is, the saturated probability space does not have the d-property at all. ■

Lemma 2 *There exists an \mathcal{I}_n -measurable map $g_n^* : (I, \mathcal{I}_n, \lambda_n) \rightarrow [-1, 1]$ such that $g_n^*(t)$ is either $h_{n-1}(t)$ or $-h_{n-1}(t)$ for any $t \in I$ and it induces the uniform distribution on $[-1, 1]$.*

Proof. Before proving this Lemma, as in [26], we first define a sequence of correspondences based on h_n . Let $\Psi : (I, \mathcal{L}, \eta) \rightarrow [-1, 1]$ be a correspondence such that $\Psi(t) = \{t, -t\}$ for any $t \in I$. Now define $\Psi_n : (I, \mathcal{I}_{n-1}, \lambda_{n-1}) \rightarrow [-1, 1]$ by $\Psi_n = \Psi \circ h_{n-1}$, that is, for any $t \in I$, $\Psi_n(t) = \{h_{n-1}(t), h_{n-1}(t)\}$. In a word, Ψ_n is the associated d-correspondence of the mapping h_{n-1} . Moreover, let $\mathcal{D}_\Psi(\eta)$ be the set of induced distributions of all Lebesgue measurable selections of Ψ and $\mathcal{D}_\Psi(\lambda)$ be the set of induced distributions of all \mathcal{I} -measurable selections of Ψ . Similarly, denote by $\mathcal{D}_{\Psi_n}(\lambda_n)$ the set of induced distributions of all \mathcal{I}_n -measurable selections of Ψ_n .

Let ν^* be the uniform distribution on $[-1, 1]$. For any $n \in \mathbb{N}$, $\nu^* \in \mathcal{D}_\Psi(\lambda) = \mathcal{D}_{\Psi_n}(\lambda_n)$ and $\mathcal{D}_{\Psi_n}(\lambda_n)$ is both convex and closed; see Section 5.2 for details. That is, there is an \mathcal{I}_n -measurable map $g_n^* : (I, \mathcal{I}_n, \lambda_n) \rightarrow [-1, 1]$ such that $g_n^*(t)$ is either $h_{n-1}(t)$ or $-h_{n-1}(t)$ for any $t \in I$ and it induces the uniform distribution on $[-1, 1]$. ■

Note that Theorem 1 is proved as a direct consequence of Proposition 2.

Proof of Theorem 1. First, we show that there does not exist any pure-strategy Nash equilibrium in the KRS-like game $\Gamma_{h_{n-1}, h_{n-1}}$ for each $n \geq 0$, where

$$\Gamma_{h_{n-1}, h_{n-1}} = \{(T_i, \mathcal{I}_i, \mu_i) = (I, \mathcal{I}_{n-1}, \lambda_{n-1}), A_i = [-1, 1], u_i^n : i = 1, 2\}.$$

This follows from Proposition 1 and the fact that h_{n-1} is a witness of the d-property for the Lebesgue extension $(I, \mathcal{I}_{n-1}, \lambda_{n-1})$, as stated in Proposition 2 (2).

Next we prove the positive results for

$$\tilde{\Gamma}_{h_{n-1}, h_{n-1}} = \{(T_i, \mathcal{I}_i, \mu_i) = (I, \mathcal{I}_n, \lambda_n), A_i = [-1, 1], u_i^n : i = 1, 2\}.$$

By Lemma 2, there exists such an \mathcal{I}_n -measurable selection g_n^* such that the value of $g_n^*(t)$ is either $h_{n-1}(t)$ or $-h_{n-1}(t)$ for any $t \in I$, and the induced distribution of g_n^* is the uniform distribution $\lambda_n(g_n^*)^{-1} = \nu^*$ over $[-1, 1]$.

We complete the proof by showing that (g_n^*, g_n^*) is a pure-strategy Nash equilibrium of $\tilde{\Gamma}_n$. Indeed, the conditions in Proposition 1 are satisfied. For player 1, given player 2's strategy g_n^* with the induced distribution ν^* on $[-1, 1]$, then $w(h_{n-1}(t), \nu^*) = 0$ for any t . As a result, from the best response correspondence as in Equations (4') and (5'), at any private information t , player 1's action $g_n^*(t)$, which is either $h_{n-1}(t)$ or $-h_{n-1}(t)$, is a best response for player 1. Similarly, we can show that given player 1's strategy g_n^* , for player 2, his action $g_n^*(t)$ as specified in the strategy g_n^* at any information $t \in I$ is also the best response. Therefore, (g_n^*, g_n^*) is a pure-strategy Nash equilibrium for $\tilde{\Gamma}_n$. ■

Proof of Lemma 1. This result follows from Theorem 1 of [8]. The model in [8] is a little different with ours. In particular, each player's private information therein consists of two parts, one part is the strategy-relevant private information on which the player can choose different actions, the other part is the payoff-relevant private information which may influence the player's payoff. In our model, we work on the special case where the payoff-relevant part shrinks to a singleton set.

Next we show that there exists a mixed strategy equilibrium in our model. Theorem 1 of [8] guarantees the existence of mixed strategy equilibrium for any incomplete information game which satisfies two conditions, one is about the regularity properties about the payoff functions as described in Assumption 3, the other is about the absolute continuity about the marginal probability

measures of μ , denote this condition by (C_2) . In particular, the marginal probability measure of μ on $(\Pi_{i=1}^\ell \mathcal{T}_i, \Pi_{i=1}^\ell \mathcal{T}_i)$, denoted by μ' , is absolutely continuous with respect to $\Pi_{i=1}^\ell \mu_i$, where μ_i is the marginal probability measure of μ on $(\mathcal{T}_i, \mathcal{T}_i)$.

We finally claim that Assumption 1 implies the condition (C_2) . Therefore, the existence of the mixed strategy equilibrium follows from Theorem 1 of [8]. We complete the proof of this lemma by providing a proof of the above claim. That is, for any $S \in \Pi_{i=1}^\ell \mathcal{T}_i$ with $\Pi_{i=1}^\ell \mu_i(S) = 0$, we have $\mu'(S) = 0$. Towards this end. Recall that $T_0 = \{t_{0p} : p \in P\}$, $S_0 = \{s_{0q} : q \in Q\}$ are both countable sets, and $\mu_0(t_{0p}, s_{0q}) = \alpha_{pq} > 0$ for any p, q . Note that,

$$\mu_i = \sum_{p \in P} \sum_{q \in Q} \alpha_{pq} \mu_i^{pq}, \text{ for all } i = 1, \dots, \ell, \quad (11)$$

$$\mu' = \sum_{p \in P} \sum_{q \in Q} \alpha_{pq} \mu^{pq}. \quad (12)$$

Fix i . Since that $\Pi_{i=1}^\ell \mu_i(S) = 0$, it follows from Fubini Theorem that there exists $S' \in \bigotimes_{j \neq i} \mathcal{T}_j$ such that (1) $\Pi_{j \neq i} \mu_j(S') = 1$; and (2) for any $t_{-i} \in S'$, the t_{-i} section of S , denoted by $S_{t_{-i}} = \{t_i : (t_i, t_{-i}) \in S\}$, is \mathcal{T}_i -measurable and $\mu_i(S_{t_{-i}}) = 0$. For any $t_{-i} \in S'$, it follows from Equation (11) that $\mu_i^{pq}(S_{t_{-i}}) = 0$ for all p, q . We next show that $\Pi_{j \neq i} \mu_j^{pq}(S') = 1$ for any p, q . Indeed,

$$\begin{aligned} \Pi_{j \neq i} \mu_j(S') = 1 &\Rightarrow \left(\Pi_{j \neq i} (\sum_{p \in P} \sum_{q \in Q} \alpha_{pq} \mu_j^{pq}) \right) (S') = 1 \quad (\because (11)) \\ &\Rightarrow \underbrace{\sum_{p_j \in P, j \neq i} \cdots \sum_{q_j \in Q, j \neq i}}_{2(\ell-1) \Sigma_s} \left(\Pi_{j \neq i} \alpha_{p_j q_j} \Pi_{j \neq i} \mu_j^{p_j q_j}(S') \right) = 1 \\ &\Rightarrow \Pi_{j \neq i} \mu_j^{p_j q_j}(S') = 1, \forall p_j, q_j, j \neq i, \end{aligned} \quad (13)$$

this is because $\sum_{p_j \in P, j \neq i} \cdots \sum_{q_j \in Q, j \neq i} \Pi_{j \neq i} \alpha_{p_j q_j} = 1$ and $\Pi_{j \neq i} \mu_j^{p_j q_j}(S') \leq 1, \forall p_j, q_j, j \neq i$. Especially, let $p_j = p, q_j = q$ for all $j \neq i$, we have that $\Pi_{j \neq i} \mu_j^{pq}(S') = 1$ for all p, q . As a result, for any p, q , $\mu^{pq}(S) = \Pi_{i=1}^\ell \mu_i^{pq}(S) = \int_{S'} \mu_i^{pq}(S_{t_{-i}}) d\Pi_{j \neq i} \mu_j^{pq} = 0$, where the first equation follows from Assumption 1 and the second from Fubini Theorem. Therefore $\mu'(S) = \sum_{p, q} \alpha_{pq} \mu^{pq}(S) = 0$ by Equation (12). \blacksquare

For our next result, we borrow the following lemma from [30, Corollary 2.3].

Lemma 3 *Let (T, \mathcal{T}, μ) be a saturated probability space, and A be a Polish space associated with the Borel algebra \mathcal{B}_A . K, J are two sets with finite or countably infinite elements. For each $k \in K$, let μ_k be a finite signed measure on (T, \mathcal{T}) that is absolutely continuous with respect to μ . For each $j \in J$, let $\psi_j : A \times T \rightarrow \mathbb{R}$ is a function such that (1) for any fixed t , $\psi_j(t, \cdot)$ is continuous on A ; (2) for any a , $\psi_j(\cdot, a)$ is \mathcal{T} -measurable and there exists a μ -integrable function ϕ_j with $|\psi_j(\cdot, a)| \leq \phi_j$ for all $a \in A$. If f is a \mathcal{T} -measurable mapping from T to $\mathcal{M}(A)$, then there is a \mathcal{T} -measurable mapping g from T to A such that,*

1. $\int_T \int_A \psi_j(t, a) f(t; da) \mu_k(dt) = \int_T \psi_j(t, g(t)) \mu_k(dt)$, for all $k \in K, j \in J$;
2. $\int_T f(t)(B) \mu_k(dt) = \mu_k(g^{-1}[B])$, for all $k \in K$ and all $B \in \mathcal{B}_A$;

3. $g(t) \in \text{supp } f(t)$ for μ -almost all $t \in T$.

In what follows, we call that the map $g : T \rightarrow A$ is a *purification* of the measure-valued mapping $f : T \rightarrow \mathcal{M}(A)$.

Proof of Theorem 2. Here is a road map of our proof. We first show that for any mixed strategy profile in the game Γ satisfying the conditions in the statement, there exists a strong purification. Next, notice that a saturated probability space is naturally an atomless probability space, thus Assumption 2 holds trivially. Lemma 1 implies that there always exists a mixed-strategy Nash equilibrium in the game Γ . Finally, by the definition of strong purification (see Definition 4), a strong purification of the mixed strategy equilibrium above is a pure-strategy Nash equilibrium of the game Γ . Therefore we only need to verify the first part, i.e., there exists a strong purification for any mixed strategy profile in the game Γ .

Recall the information structure in this game is $(\Omega, \mathcal{F}, \mu) = (S_0 \times \prod_{j=0}^{\ell} T_j, \mathcal{S}_0 \otimes \prod_{j=0}^{\ell} \mathcal{T}_j, \mu)$. For any $p \in P$ and $q \in Q$, $\mu_0(\{(t_{0p}, s_{0q})\}) = \mu(\{(t_{0p}, s_{0q})\} \times T) = \alpha_{pq} > 0$. According to the conditionally independence, Assumption 1, we have $\mu^{pq} = \mu(\cdot | t_{0p}, s_{0q}) = \prod_{i=1}^{\ell} \mu_i^{pq}$ for each $p \in P, q \in Q$. As a result, for any player i and any \mathcal{T}_i -measurable subset S_i , $\mu_i(S_i) = \sum_{p \in P} \sum_{q \in Q} \alpha_{pq} \mu_i^{pq}(S_i)$. Thus each μ_i^{pq} is absolutely continuous with respect to μ_i .

Assume that $f = (f_1, \dots, f_{\ell})$ is a mixed strategy profile of the game Γ . That is, for any player i , f_i is a measurable mapping from $T_0 \times T_i$ to $\mathcal{M}(A_i)$, the space of Borel probability measures on the action space A_i . Fix player i . Her expected payoff $U_i(f)$ in this mixed strategy profile f as defined in Equation (9) can be written as follows, note that $T = \prod_{j=1}^{\ell} T_j, A = \prod_{j=1}^{\ell} A_j$,

$$\begin{aligned} U_i(f) &= \int_{\Omega} \int_A u_i(a, s_0, t_i) f_1(t_0, t_1; da_1) \cdots f_{\ell}(t_0, t_{\ell}; da_{\ell}) d\mu(\omega) \\ &= \sum_{p \in P} \sum_{q \in Q} \alpha_{pq} \int_T \int_A u_i(s_{0q}, t_i, a) \prod_{j=1}^{\ell} f_j(t_{0p}, t_j; da_j) d\mu^{pq} \\ &= \sum_{p \in P} \sum_{q \in Q} \alpha_{pq} \int_T \int_A u_i(s_{0q}, t_i, a) \prod_{j=1}^{\ell} f_j(t_{0p}, t_j; da_j) \prod_{j=1}^{\ell} d\mu_j^{pq}(t_j) \\ &= \sum_{p \in P} \sum_{q \in Q} \alpha_{pq} \int_{T_i} \int_{A_i} v_{ipq}^f(t_i, a_i) f_i(t_{0p}, t_i; da_i) \mu_i^{pq}(dt_i), \end{aligned} \quad (14)$$

where

$$v_{ipq}^f(t_i, a_i) = \int_{T_{-i}} \int_{A_{-i}} u_i(s_{0q}, t_i, a_i, a_{-i}) \prod_{j \neq i} f_j(t_{0p}, t_j; da_j) \prod_{j \neq i} d\mu_j^{pq}(t_j). \quad (15)$$

Conditional on the event $t_0 = t_{0p}, s_0 = s_{0q}$, let $\gamma_{j pq}^{f_j}$ be the induced probability distribution of f_j on A_j for player $j, j = 1, \dots, \ell$. That is, for any Borel subset B_j of A_j ,

$$\gamma_{j pq}^{f_j}(B_j) = \int_{T_j} f_j(t_{0p}, t_j)(B_j) d\mu_j^{pq}(t_j). \quad (16)$$

Consequently, we have the following equation,

$$v_{ipq}^f(t_i, a_i) = \int_{A_{-i}} u_i(s_{0q}, t_i, a_i, a_{-i}) d\prod_{j \neq i} \gamma_{j pq}^{f_j}. \quad (17)$$

Equations (14) and (17) imply that, conditional on the event $t_0 = t_{0p}, s_0 = s_{0q}$, player i 's expected payoff depends on the actions of the other players only through $\gamma_{jppq}^{f_j}, j \neq i$, the induced conditional distributions of their strategies on their action spaces.

For the fixed player i and the fixed mixed strategy profile f , since player i 's payoff function u_i satisfies Assumption 3, it is clear that for each p, q , $v_{ipq}^f(t_i, a_i)$ defined in Equation (15) satisfies the following conditions: (1) for any $t_i \in T_i$, $v_{ipq}^f(t_i, \cdot)$ is continuous in A_i , (2) for any $a_i \in A_i$, $v_{ipq}^f(\cdot, a_i)$ is \mathcal{T}_i -measurable on T_i , and (3) $|v_{ipq}^f(t_i, a_i)| \leq \int_{T_{-i}} \phi_i(s_{0q}, t_{0p}, t_i, t_{-i}) \Pi_{j \neq i} d\mu_j^{pq}(t_j)$ since $|u_i(a, s_0, t_i)| \leq \phi_i(s_0, t_0, t_i, t_{-i})$ holds for each $a \in A$. Therefore, we can apply Lemma 3 to the following parameters to obtain a purification for the measure-valued mapping $f_i(t_{0p}, \cdot)$ for each $p \in P$,

$$\left[(T_i, \mathcal{T}_i), \{\mu_i^{pq}\}_{q \in Q}, A_i, \{v_{ipq}^f\}_{q \in Q}, f_i(t_{0p}, \cdot) \right]$$

In particular, for any $p \in P$, there exists a \mathcal{T}_i -measurable map $g_i^p : T_i \rightarrow A_i$ with the following properties,

- (i) $\int_{T_i} \int_{A_i} v_{ipq}^f(t_i, a_i) f_i(t_{0p}, t_i; da_i) \mu_i^{pq}(dt_i) = \int_{T_i} v_{ipq}^f[t_i, g_i^p(t_i)] \mu_i^{pq}(dt_i)$, for any $q \in Q$;
- (ii) for any Borel set $B_i \subseteq A_i$ and $q \in Q$, $\gamma_{ipq}^{f_i}(B_i) = \int_{T_i} f_i(t_{0p}, t_i)(B_i) \mu_i^{pq}(dt_i) = \mu_i^{pq}((g_i^p)^{-1}[B_i])$;
- (iii) $g_i^p(t_i) \in \text{supp } f_i(t_{0p}, t_i)$ for μ_i -almost all $t_i \in T_i$.

In the above arguments, since the choice of i and p are arbitrarily, then the arguments can be applied for all $i = 1, \dots, \ell$ and $p \in P$. Fix $g_i^p : T_i \rightarrow A_i$ to be a purification with the above properties (i)-(iii) of the measure-valued mapping $f_i(t_{0p}, \cdot) : T_i \rightarrow \mathcal{M}(A_i)$ for each $i, p \in P$. Denote by $g_i : T_0 \times T_i \rightarrow A_i$ the map defined as $g_i(t_{0p}, t_i) = g_i^p(t_i)$ for any $p \in P$. Note that it is a pure strategy for player i . Let $g = (g_1, \dots, g_\ell)$, it is clear a pure-strategy profile.

Finally, we claim that the above pure-strategy profile g is a strong purification of mixed strategy profile f . That is, it satisfies Items 1-4 in Definition 4. It is clear that the Items 3 and 4 are the above Assertions (ii) and (iii) respectively. We only need to prove Items 1 and 2 in the definition. Towards this end, for any mixed strategy \tilde{f}_i of player i , let $\tilde{f} = (\tilde{f}_i, f_{-i})$, and $\tilde{g} = (\tilde{f}_i, g_{-i})$. By Equation (17), the expected payoff of player i with \tilde{f} and \tilde{g} are respectively given by

$$U_i(\tilde{f}) = \sum_{p \in P} \sum_{q \in Q} \alpha_{pq} \int_{T_i} \int_{A_i} v_{ipq}^{\tilde{f}}(t_i, a_i) \tilde{f}_i(t_{0p}, t_i; da_i) \mu_i^{pq}(dt_i), \quad (18)$$

$$U_i(\tilde{g}) = \sum_{p \in P} \sum_{q \in Q} \alpha_{pq} \int_{T_i} \int_{A_i} v_{ipq}^{\tilde{g}}(t_i, a_i) \tilde{f}_i(t_{0p}, t_i; da_i) \mu_i^{pq}(dt_i). \quad (19)$$

By the definition of the strategy profiles \tilde{f} and \tilde{g} , $\tilde{f}_{-i} = f_{-i}$ and $\tilde{g}_{-i} = g_{-i}$. As a result, for any $j \neq i, p \in P$ and $q \in Q$, by the definition of $\gamma_{jppq}^{f_j}$ in Equation (16), $\gamma_{jppq}^{\tilde{f}_j} = \gamma_{jppq}^{f_j}$ and $\gamma_{jppq}^{\tilde{g}_j} = \gamma_{jppq}^{g_j}$. By Assertion (ii) above $\gamma_{jppq}^{\tilde{f}_j} = \gamma_{jppq}^{g_j}$ for all $j \neq i, p$ and q . Hence by the definition of v_{ipq}^f in Equation (15), $v_{ipq}^{\tilde{f}}(t_i, a_i) = v_{ipq}^f(t_i, a_i) = v_{ipq}^g(t_i, a_i) = v_{ipq}^{\tilde{g}}(t_i, a_i)$. Therefore, it follows by

Equations (18) and (19) that $U_i(\tilde{f}) = U_i(\tilde{g})$. We thus proved Item 2 in Definition 4. The properties in Item 1 can also be obtained in a similar way. In fact,

$$\begin{aligned} U_i(g) &= \sum_{p \in P} \sum_{q \in Q} \alpha_{pq} \int_{T_i} v_{ipq}^g(t_i, g_i^p(t_i)) \mu_i^{pq}(dt_i) \\ &= \sum_{p \in P} \sum_{q \in Q} \alpha_{pq} \int_{T_i} v_{ipq}^f(t_i, g_i^p(t_i)) \mu_i^{pq}(dt_i) \\ &= U_i(f), \end{aligned}$$

where the second equation follows from Assertion (ii) above. Therefore we complete the proof that the pure-strategy profile g is a strong purification of the mixed strategy profile f . \blacksquare

Proof of Theorem 3. Without loss of generality, suppose that the private information space for player 1, $(T_1, \mathcal{T}_1, \mu_1)$, is an atomless but not a saturated probability space. By Part (1) of Proposition 2, it is a space satisfying the d-property. As mentioned in Remark 2, to construct a ℓ -player game with the given information structure and the uncountable compact matrix spaces A_i but without pure-strategy Nash equilibrium, one need only consider a simpler game with two players, players 1 and 2 and with $A_1 = A_2 = [-1, 1]$.

Because $(T_1, \mathcal{T}_1, \mu_1)$ is a space satisfying the d-property, let h_1 be one witness map of the d-property for the space and the subspace restricted to $h_1^{-1}([0, 1/2])$. Meanwhile, since $(T_2, \mathcal{T}_2, \mu_2)$ is an atomless probability space, it is a known result that there always exists a \mathcal{T}_2 -measurable mapping h_2 from $(T_2, \mathcal{T}_2, \mu_2)$ to the Lebesgue interval such that it induces the Lebesgue measure on the interval.

Now we consider the KRS-like game Γ_{h_1, h_2} ,

$$\Gamma_{h_1, h_2} = \{(T_i, \mathcal{T}_i, \mu_i), A_i = [-1, 1], \tilde{u}_i : i = 1, 2\},$$

where \tilde{u}_i is defined in Equations (6) and (7). It is clear that the payoff functions $\tilde{u}_i, i = 1, 2, \dots, \ell$ satisfy Assumption 3 since $u_i, i = 1, 2$ defined in Equations (1) and (2) satisfy this Assumption and h_i are \mathcal{T}_i -measurable mappings.

Suppose that there exists a pure-strategy Nash equilibrium (g_1^*, g_2^*) for the KRS-like game Γ_{h_1, h_2} . Then by the best response correspondences as in Equations (4') and (5'), g_1^* must be a selection of the d-correspondence $\{h_1(t_1), -h_1(t_1)\}$, and the induced distribution of g_1^* , when restricted to $h_1^{-1}(0, 1/2)$ is uniform on $[-1/2, 1/2]$. This contradicts the fact that h_1 is a witness of the d-property of $(T_1, \mathcal{T}_1, \mu_1)$. \blacksquare

Proof of Theorem 4. Suppose that for some $t_{0p} \in T_0$ and $s_{0q} \in S_0$, $(T_i, \mathcal{T}_i, \mu_i^{pq})$ is a not saturated probability space for some player i . In the information structure $(\Omega, \mathcal{F}, \mu)$, for $t_0 = t_{0p}, s_0 = s_{0q}$, recall that $\alpha_{pq} = \mu_0(\{(t_{0p}, s_{0q})\}) > 0$ and μ^{pq} is the conditional probability measure of μ on $(\prod_{j=1}^{\ell} T_j, \prod_{j=1}^{\ell} \mathcal{T}_j)$ when $t_0 = t_{0p}, s_0 = s_{0q}$.

Now consider a private information game Γ with the above information structure and the payoff functions and for any $q' \neq q$,

$$u_i(s_{0q'}, a, t_i) \equiv 0, \text{ for all } a = (a_i, a_{-i}) \in A, t_i \in T_i, \text{ and player } i. \quad (20)$$

Notice that the payoff functions specified in Equation (20) for other $q' \neq q$ satisfy the conditions in Assumption 3 trivially. Moreover, if $u_i(s_{0q}, \cdot, \cdot)$ for any player i also satisfies the conditions in Assumption 3, then the payoff structure in game Γ has the properties specified in Assumption 3. Thus, Assumptions 1-3 are satisfied in game Γ in this case. As a consequence, if for each player i , $u_i(s_{0q}, \cdot, \cdot)$ satisfies the conditions in Assumption 3, it follows from the assumption of this theorem that there exists a pure-strategy Nash equilibrium in game Γ .

We next characterize the pure-strategy Nash equilibrium in the game Γ above. Assume that $g^* = (g_1^*, \dots, g_\ell^*)$ is pure-strategy Nash equilibrium in Γ , where for each player i , $g_i^* : T_0 \times T_i \rightarrow A_i$ is a $\mathcal{T}_0 \otimes \mathcal{T}_i$ -measurable mapping. For simplicity, denote $g_i^*(t_{0p'}, \cdot)$ by $g_i^{*p'}$ for each $p' \in P$ and write g_i^* as (g_i^{*p}, g_i^{*-p}) . By Equation (9), the expected payoff for player i in the pure-strategy equilibrium g^* is,

$$\begin{aligned} U_i(g^*) &= U_i(g_i^*, g_{-i}^*) = \sum_{p' \in P} \sum_{q' \in Q} \alpha_{p'q'} \int_T u_i(s_{0q'}, t_i, g_i^{*p'}(t_i), g_{-i}^{*p'}(t_{-i})) \, d\mu^{p'q'} \\ &= \sum_{p' \in P} \alpha_{p'q} \int_T u_i(s_{0q}, t_i, g_i^{*p'}(t_i), g_{-i}^{*p'}(t_{-i})) \, d\mu^{p'q}. \end{aligned} \quad (21)$$

where the last equation holds because of Equation (20). By the definition of pure-strategy equilibria, for any player i , for any pure-strategy $\bar{g}_i : T_0 \times T_i \rightarrow A_i$, we have $U_i(g_i^*, g_{-i}^*) \geq U_i(\bar{g}_i, g_{-i}^*)$; that is,

$$\sum_{p' \in P} \alpha_{p'q} \int_T u_i(s_{0q}, t_i, g_i^{*p'}(t_i), g_{-i}^{*p'}(t_{-i})) \, d\mu^{p'q} \geq \sum_{p' \in P} \alpha_{p'q} \int_T u_i(s_{0q}, t_i, \bar{g}_i^{p'}(t_i), g_{-i}^{*p'}(t_{-i})) \, d\mu^{p'q}. \quad (22)$$

Given game Γ and the fixed $p \in P, q \in Q$, let Γ^{pq} be the following private information game with a single pair of common states t_{0p}, s_{0q} ,

$$\Gamma^{pq} = \{(T_i, \mathcal{T}_i), A_i, u_i(s_{0q}, \cdot), i = 1, \dots, \ell, \mu^{pq}\}.$$

According to Assumption 1, $\mu^{pq} = \prod_{i=1}^\ell \mu_i^{pq}$. As a result, the private information structure for the game Γ^{pq} is the product probability space $(\prod_{i=1}^\ell T_i, \prod_{i=1}^\ell \mathcal{T}_i, \prod_{i=1}^\ell \mu_i^{pq})$.

The relation between the pure-strategy Nash equilibria in the games Γ and Γ^{pq} is as follows. If there exists a pure-strategy Nash equilibrium in the game Γ , then there exists a pure-strategy Nash equilibrium in game Γ^{pq} . In particular, assume that $g^* = (g_1^*, \dots, g_\ell^*)$ is a pure-strategy Nash equilibrium in Γ , then $g^{*p} = (g_1^{*p}, \dots, g_\ell^{*p})$ is a pure-strategy Nash equilibrium in game Γ^{pq} . In fact, for each player i and any pure-strategy $g_i^p : T_i \rightarrow A_i$, let $g_i' = (g_i^p, g_i^{*-p})$. It is clear that g_i' is a pure-strategy for player i in game Γ . Applying the inequalities in (22) by replacing player i 's strategy by g_i' in the right hand side, notice that $\alpha_{pq} > 0$, we have that

$$\int_T u_i(s_{0q}, t_i, g_i^{*p}(t_i), g_{-i}^{*p}(t_{-i})) \, d\mu^{pq} \geq \int_T u_i(s_{0q}, t_i, g_i^p(t_i), g_{-i}^{*p}(t_{-i})) \, d\mu^{pq}.$$

Since g_i^p is an arbitrary pure-strategy of an arbitrary player i in game Γ^{pq} , therefore the pure-strategy profile $g^{*p} = (g_1^{*p}, \dots, g_\ell^{*p})$ is a pure-strategy Nash equilibrium for game Γ^{pq} .

We are now ready to complete the proof of this theorem. Take the following terms as given, a finite set of ℓ players, two countable sets S_0 and T_0 , uncountable compact metric spaces A_i , and

the private information structure $(\Omega, \mathcal{F}, \mu)$ which satisfies Assumptions 1 and 2. Let Γ^{pq} be a private information game with the private information structure $(T, \mathcal{T}, \mu^{pq} = \prod_{i=1}^{\ell} \mu_i^{pq})$ and action spaces A_i , $i = 1, \dots, \ell$, suppose that the payoff functions u_i satisfies Assumption 3. Then we can extend Γ^{pq} to be a private information game Γ' with the private information structure $(\Omega, \mathcal{F}, \mu)$ by defining the payoff functions at other $t_{0p'}, p' \neq p$ as in Equation (20). It is clear that this game Γ' satisfies Assumptions 1-3, and it follows from the hypotheses of the theorem that there always exists a pure-strategy Nash equilibrium in Γ' . Moreover, due to the arguments above, there also exists a pure-strategy Nash equilibrium in the game Γ^{pq} . Then, according to Theorem 3, the private information space of player i , $(T_i, \mathcal{T}_i, \mu_i^{pq})$, must be a saturated probability space. Thus we complete the proof of this theorem. ■

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